Journal of Finance & Economics
Volume 1, Issue 4 (2013), 01-09
ISSN 2291-4951 E-ISSN 2291-496X
Published by Science and Education Centre of North America

Numerical Aspects to Estimate the Generalized Hyperbolic Probability Distribution

José Antonio Núñez Mora[1*], Leovardo Mata Mata[1]

[1]Tecnologico de Monterrey, Mexico

*Correspondence: José Antonio Núñez Mora. Calle del Puente 222, Col. Ejidos de Huipuilco 14380 Tlalpan, Mexico, D. F.

Abstract: In this paper we present numerical aspects of a modified EM algorithm of maximum likelihood to estimate the generalized hyperbolic probability distribution. The estimation is considering the parameter λ of the modified Bessel function of third order as a no constrained parameter. A suitable starting point for the numerical method is proposed to reduce the number of iterations. The goodness of fit is valuated with the log-likelihood function through an empirical test for multivariate distributions.

JEL Classifications: G15, C40

Keywords: EM-algorithm, Generalized Hyperbolic distribution, multivariate modeling

1. Introduction

A lot of evidence in the academic literature has demonstrated that the distributions of the financial returns in the stock market are not normal, see for example Eberlein and Keller (1995) and for the Mexican case, Trejo Becerril, Lorenzo Valdés, and Núñez Mora (2006). However, numerical difficulties emerge in the case of non-normal distributions and therefore the applicability in important economic and financial phenomena can be limited. In particular, problems can appear in the case of the family of generalized hyperbolic distributions (GH), because of the number of parameters. On the other hand, the number of parameters represents a great versatility.

Barndorff-Nielsen (1977) began to use the GH family in the study of the sand grains and some years were necessary to obtain robust numerical implementations. An outstanding progress in the applications of the generalized hyperbolic family was made by Blæsild and Sørensen (1992) with the development of the program HYP.

The program HYP adjusts hyperbolic distributions using maximum likelihood but the number of dimensions was less or equal than three. Prause (1999) analyzed exhaustively different specifications of calibrations for the log-likelihood function of the GH family through HYP with at most dimension three.

Eberlein and Keller (1995) adjusted the univariate hyperbolic distribution for different returns of the German Stock Exchange with high accuracy. However, the multivariate case was not reported because there were problems with the convergence and the computing time using maximum likelihood.

The solution for the estimation problem appeared with the EM (expectation-maximization) algorithm. This powerful procedure presented by Dempster, Laird, and Rubin (1977) can be used in maximum likelihood estimations; for example, Liu and Rubin (1995) made use of this algorithm with the extensions ECME (expectation conditional maximization) and MCECM (multicycle expectation conditional maximization) to calibrate the student t distribution.

An important modification was developed by Protassov (2004) in order to estimate generalized hyperbolic multivariate distributions. The parameter λ is fixed in the Bessel function of third kind in the augmented maximum likelihood function. The article of Protassov is the first one reporting an estimation of the GH family for dimension greater than three. In particular, the Normal Inverse Gaussian distribution (λ=-1/2) of dimension five for different exchange rate returns.

This paper uses the parametrization of Protassov (2004) together with observations made by McNeil, Frey, and Embrechts (2005) and Hu (2005) to extend and modify the estimation procedure of the multivariate GH family.

Hu (2005) analyzed thoroughly the limit cases of the GH family and discovered that the algorithm of Protassov is numerically unstable for little values|λ|<1 and combined the specification of McNeil *et al.* (2005) together the procedure of Protassov to estimate the limit cases of the GH family: the t-student, Variance Gamma and the Normal Inverse Gaussian (NIG) distributions. At the same time the number of iterations and the computing time are reduced for |λ|<10. In this last case, it was convenient to fix the parameters ψ and χ of the associated Gaussian Inverse Generalized function (GIG). This last fact contrasted with the work of Protassov, where only the parameters of the NIG distribution are estimated.

Different parametrizations are developed by Lüthi (2011), Scott (2012) and R Development Core Team (2011) to estimate members of the GH family. The estimation for the univariate case is obtained by maximum likelihood, like in Prause (1999) and Blæsild and Sørensen (1992) and for the multivariate case the procedure used is MCECM, see example Meng and Rubin (1993) and McNeil *et al.* (2005). In this last procedure the parameter λ is permitted to be free and is calculated together the other parameters. However, Breymann and Lüthi (2011) did not choose systematically an initial point for the numerical algorithm. A suitable election of the initial point permit us to get a much better estimation of the parameters, reduce the number of iterations necessary to converge and avoid complications related to the instability of the procedure when we have certain values of λ.

In this paper a modification and extension of the EM algorithm of Protassov (2004) is presented in order to estimate the parameters of the GH family. We do not assume that the parameter λ is fixed like in Protassov (2004). At the same time an initial point for the numerical procedure is taken for the parameters of the distribution. The choice of this point under the GIG distribution uses results about invariance of the parameters in the linear combination of the GH. The goodness of fit is studied trough an empirical test for multivariate distributions, see McAssey (2013). This last one is a modification of the existent methods to facilitate a measure of the adjustment of the GH distributions and its potential application in different fields.

2. Generalized Hyperbolic Distribution

The family of generalized hyperbolic distributions can be obtained mixing a multivariate normal distribution and a Generalized Inverse Gaussian distribution (GIG).

This specification is established in terms of the parameters λ, χ, y, ψ of the GIG distribution, the parameters of the normal multivariate distribution μ, y, Σ and a vector of biasγ.

The GIG density function is given by

$$f(x;\lambda,\chi,\psi) = \frac{1}{k_\lambda(\chi\psi)} x^{\lambda-1} \exp\left[-\frac{1}{2}(\chi x^{-1} + \psi x)\right] \tag{1}$$

where $k_\lambda(\chi,\psi)$is another specification of the third kind Bessel function, see Paolella (2007) and is given by

$$k_\lambda(\chi,\psi) = \int_0^\infty x^{\lambda-1} \exp\left[-\frac{1}{2}(\chi x^{-1} + \psi x)\right] dx, \tag{2}$$

where λ is a real number and $\chi,\psi \geq 0$, Abramowitz and Stegun (1972). Through GIG distribution it can be established the GH mixing mean-variance for a $n \times 1$ random vectorX

$$X|W = w \sim N_n(\mu + w\beta\Delta, w\Delta) \tag{3}$$
$$W \sim GIG(\lambda,\chi,\psi)$$

Usually the new parametrization of the GIG density function through the parameters $\alpha,\delta \geq 0$ such that$\chi = \delta^2$ and $\psi = \alpha^2 - \beta'\Delta\beta$.

The multivariate GH density like in Protassov (2004) for a random vector X of dimension $n \times 1$ is given by

$$f(x;\lambda,\gamma,\beta,\mu,\delta,\Delta) = \frac{\left(\sqrt{\frac{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)'}{\gamma^2+\beta'\Delta\beta}}\right)^{\lambda-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}\left(\frac{\gamma}{\delta}\right)^{-\lambda}} \frac{K_{\lambda-\frac{n}{2}}\left(\sqrt{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)'(\gamma^2+\beta'\Delta\beta)}\right)}{K_\lambda(\delta\gamma)\exp\left(-(x-\mu)\beta'\right)} \tag{4}$$

Where the modified third order Bessel function is

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty w^{\lambda-1}\exp\left(\left[-\frac{1}{2}x(w+w^{-1})\right] dw\right. \tag{5}$$

for $x > 0$.

There are different parameterizations for the multivariate GH distribution and in this paper we consider the specification presented by Protassov (2004), where the problem of specification can be solved assigning the determinant of Δ equal to one. In this case, the univariate density function is defined as

$$f(x) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\sqrt{\delta^2+(x-\mu)^2}\right)^{\frac{1}{2}-\lambda}} \exp\left(\beta(x-\mu)\right) \tag{6}$$

The univariate density function is very important in GH family in the proposition presented by McNeil *et al.* (2005). In that proposition is established that if a random vector X of dimension $n \times 1$ ismultivariate generalized hyperbolic distributed with parameters $(\lambda,\chi,\psi,\mu,\Sigma,\gamma)$, then if we define $Y = BX + b$ where B is matrix $k \times n$ and b a $k \times 1$ vector, then Y is hyperbolic with parameters $(\lambda,\chi,\psi,B\mu + b, B\Sigma B', B\gamma)$.

In particular, if $b = 0$ and $k = 1$ the linear combinationYof the elements of the random vectorXhas a univariate hyperbolic distribution. In this case the parameters (λ,χ,ψ)associated with the GIG in the mean-variance mixing are invariant.

This result permits us to suggest in a systematic way, an initial point for the use of the EM algorithm in the estimation of the parameters in the multivariate GH distribution.

Given the invariance of the parameters (λ,χ,ψ), we use a linear combination distributed as a hyperbolic univariate distribution and the estimation of the optimal parameters $(\lambda^0, \chi^0, \psi^0)$ is trough maximum likelihood. This estimation is used as an initial point for the numerical procedure of the EM algorithm.

3. Estimation Procedure

Let us take a sample of random vectors $X_1, X_2, \dots, X_m$from a multivariate generalized hyperbolic population with probability density function determined by the parameters$\Theta = (\lambda, \gamma, \delta, \beta, \mu, \Delta)$, the log-likelihood function is

$$\ln\left(L(\Theta; x_1, \dots, x_m\right) = \sum_{i=1}^{m} \ln\left(f(x_i; \Theta)\right) \tag{7}$$

The key point of the EM algorithm is that it introduces latent variables $w_1, w_2, \dots, w_m$, whose distribution GIG is such that defines an augmented likelihood function

$$\ln\Big(L^{'}(\Theta;\ x_1, \dots, x_m, w_1, w_2, \dots, w_m)\Big) = \sum_{i=1}^{m} \ln\left(f_{X|W}(x_i|w_i; \mu, \Delta, \beta)\right) + \sum_{i=1}^{m} \ln\left(f_W(w_i; \lambda, \gamma, \delta\right) \tag{8}$$

Given an initial approximation Θ_0, the EM algorithm is a recursive method which consists of two steps:

Step 1. Calculate the expected value of the augmented log likelihood function by the starting point Θ_0 and the random sample$x_1, x_2, \dots, x_m$. Thus each kth iteration obtain the objective function

$$g\left(\Theta; \Theta^{[k]}\right) = E\left[\ln\left(L^{'}(\Theta;\ x_1, \dots, x_m, w_1, w_2, \dots, w_m)\right) | x_1, \dots, x_m; \Theta^{[k]}\right] \tag{9}$$

Step 2. Maximize the objective function $g\left(\Theta; \Theta^{[k]}\right)$ to obtain the following approximation $\Theta^{[k+1]}$.

In our case, the choice of our approach considers the invariance property in the GH family of linear combinations, see McNeil *et al.* (2005) and robustness of the median as a measure of central tendency when we have that the empirical distribution is not symmetric, Paolella (2007). In particular, the invariance property holds for the canonical basis of $\mathbb{R}^n$. Therefore we take as a starting point the median of the maximum likelihood univariate function itself (not from the expected value of such function). Estimators are found by univariate linear combinations $Y = B_j X$, $j = 1,2, \dots, n$, where B_j vectors are the standard basis in $\mathbb{R}^n$; and X is the data matrix with m rows and n columns. Initial parameter $\left(\lambda^0, \delta^0, \gamma^0\right)$ reduce the number of iterations, where$\delta^0 = \sqrt{\chi^0}$ and $\gamma^0 = \sqrt{\psi^0}$, because Y follows an univariate generalized hyperbolic distribution, see McNeil *et al.* (2005), with the same parameters λ, χ, ψ of the original multivariate distribution. The univariate density function specification we used is indicated by Paolella (2007). There is no problem with the limiting cases, i.e. when χ or ψ tend to zero.

We make the change of variable like in Protassov (2004),

$$\begin{aligned} \tilde{\mu} &= \mu \\ \tilde{\gamma} &= \gamma\delta \\ \tilde{\Sigma} &= \delta^2\Delta \\ \tilde{\beta} &= \delta^2\beta\Delta \end{aligned} \tag{10}$$

Next $w_i|x_i,\Theta^{[k-1]} \sim GIG(\lambda^*_{k-1},\delta^*_i,\gamma^*_{k-1})$ for $i = 1,2,\dots,m$ with $\lambda^*_{k-1} = \lambda_{k-1} - n/2$ which implies that

$$\delta^*_i = \sqrt{1 + (x_i - \tilde{\mu}_{k-1})\left[\tilde{\Sigma}^{-1}_{k-1}\right](x_i - \tilde{\mu}_{k-1})'} \tag{11}$$

$$\gamma^*_{k-1} = \sqrt{(\tilde{\gamma}_{k-1})^2 + (\tilde{\beta}_{k-1})\left[\tilde{\Sigma}^{-1}_{k-1}\right](\tilde{\beta}_{k-1})'}$$

and the corresponding conditional moments are calculated according to $GIG(\lambda^*_k,\delta^*_i,\gamma^*_k)$:

$$E\left[w_i|x_i,\Theta^{[k-1]}\right] = \frac{\delta^*_i}{\gamma^*_k}\frac{K_{\lambda^*_k+1}(\delta^*_i\gamma^*_k)}{K_{\lambda^*_k}(\delta^*_i\gamma^*_k)} \tag{12}$$

$$E\left[w_i^{-1}|x_i,\Theta^{[k-1]}\right] = \frac{\gamma^*_k}{\delta^*_i}\frac{K_{\lambda^*_k-1}(\delta^*_i\gamma^*_k)}{K_{\lambda^*_k}(\delta^*_i\gamma^*_k)}$$

Then

$$\tilde{\Sigma}_k = \left(\frac{1}{m}\right)\sum_{i=1}^{m}\begin{pmatrix}(x_i - \tilde{\mu}_k)^{'(x_i-\tilde{\mu}_k)}E\left[w_i^{-1}|x_i,\Theta^{[k-1]}\right] - (\tilde{\beta}_k)^{'(x_i-\tilde{\mu}_k)} \\ -(x_i - \tilde{\mu}_k)'(\tilde{\beta}_k) + (\tilde{\beta}_k)'(\tilde{\beta}_k)E\left[w_i|x_i,\Theta^{[k-1]}\right]\end{pmatrix} \tag{13}$$

Maximize the function

$$m\lambda_{k-1}ln(\tilde{\gamma}_{k-1}) - mln\left(K_{\lambda_{k-1}}(\tilde{\gamma}_{k-1})\right) - \left(\frac{\tilde{\gamma}_{k-1}{}^2}{2}\right)\sum_{i=1}^{m}E\left[w_i|x_i,\Theta^{[k-1]}\right] \tag{14}$$

and update λ_{k-1} and $\tilde{\gamma}_{k-1}$. Finally calculate

$$\begin{aligned} \mu &= \tilde{\mu} \\ \delta &= \left|\tilde{\Sigma}\right|^{1/2n} \\ \gamma &= \tilde{\gamma}/\delta \\ \beta &= \tilde{\beta}\tilde{\Sigma}^{-1} \\ \Delta &= \frac{1}{\left|\tilde{\Sigma}\right|^{1/n}}\tilde{\Sigma} \end{aligned} \tag{15}$$

4.1 An Empirical Test for Multivariate Distribution GH

In general, the issue of goodness of fit for multivariate distributions is scarce. There are extensive studies to the multivariate normal case, but little has been said of the general case. There are proposals to extend the Kolmogorov-Smirnov test to the case n dimensional, Loudin and Hannu (2003) and Justel, Peña, and Zamar (1997), but application and implementation for dimensions greater than two is not simple in general cases. Similarly, there are proposals of nonparametric multivariate kernel estimates, see Gábor and Rizzo (2004), but also the same problem appears in the implementation into higher dimensions.

In this paper we incorporate a simple test of McAssey (2013) that recently appeared in the literature for continuous multivariate distributions of any dimension. Using this procedure testing the goodness of fit to the multivariate generalized hyperbolic distribution and reinforcing the estimation and inference of our estimations.

In summary, the algorithm described by McAssey (2013) in our context is as follows:

a) We assume that the null hypothesis is true, i.e., we assume that the series of exchange rate returns follow a multivariate generalized hyperbolic distribution of dimension five and via the EM algorithm estimate the corresponding parameters.

b) We generate a sample $u_1, u_2, \dots, u_N$ of size 10000 by simulation with the parameters of subsection a). Simulation of GH family uses the algorithm in Dagpunar (1989), Kinderman (1977) and the condition $X|W = w \sim N_n(\mu + w\beta\Delta, w\Delta)$ with $W \sim GIG(\lambda, \chi, \psi)$.

c) We calculate the Mahalanobis distance for the series of subsection b) using the mean $\hat{\mu}$ and variance $\hat{\Sigma}$ sample:

$$\hat{d}_i = \sqrt{(\hat{u}_i - \hat{\mu})\hat{\Sigma}^{-1}(\hat{u}_i - \hat{\mu})}. \tag{16}$$

d) Next for $t \in \left(0, 2m\acute{a}x\{\hat{d}_i\}\right)$ calculate

$$\hat{G}_N(t) = \frac{1}{N}\sum_{i=1}^{N} I\left(\hat{d}_i \leq t\right). \tag{17}$$

e) Choose a partition $\{p_0, p_1, \dots, p_T\}$ in $[0,1]$ with $p_0 = 0$, $p_T = 1$, and calculate $q_0 = 0$, $q_j = min\{t \in \mathbb{R} | \hat{G}_N(t) \geq p_j\}$ for $j = 1,2, \dots, T-1$ y $q_T = \infty$.

f) Finally estimate $E_j = n\left(p_j - p_{j-1}\right)$ and count O_j observations $\hat{d}_i$ in $(q_{j-1}, q_j]$ for $j = 1,2, \dots, T-1$ and $(q_{T-1}, \infty]$. Next the statistic is

$$A_T = \sum_{j=1}^{T} \frac{|E_j - O_j|}{E_j} \tag{18}$$

Find the p-value using a simulation sample of size M on adjusted multivariate distribution function, McAssey (2013), and we reject the null hypothesis if p-value is less than some given level of significance.

5. Results

We use daily quotes on exchange rates (mid-price) of five currencies versus the American dollar from Bloomberg. The five currencies are the Swiss franc, the Deutschemark, the British pound, the Canadian dollar and the Japanese yen. The time period is 7/2/1986 through 10/31/2001 like in Protasov (2004). We calculate the exchange rates into logarithmic returns series. Tables 1 and 2 show the estimations with $\lambda = -0.5$ versus free λ and systematic starting point.

Table 1.

Estimation via the EM algorithm with λ= -0.5					
Iterations	1071				
p-value	0.112				
Log-likelihood	79948.62				
λ	-0.500				
δ	0.005				
γ	272.434				
Matrix Δ					
Currency	CHF	DEM	GBP	CAD	JPY
CHF	2.84577	2.37786	1.51159	0.00970	1.26953
DEM	2.37786	2.34835	1.40543	0.02615	1.12566
GBP	1.51159	1.40543	1.96464	0.03819	0.78977
CAD	0.00970	0.02615	0.03819	0.46850	-0.01940
JPY	1.26953	1.12566	0.78977	-0.01940	2.47008
Currency	CHF	DEM	GBP	CAD	JPY
β	36.3603	-25.7984	-17.2824	-9.0025	10.4371
μ	-0.0005	-0.0003	0.0001	0.0001	-0.0005

Table 1 presents the estimates considering the fixed parameter $\lambda = -0.5$ and taking the initial point $\tilde{\gamma} = 1$, $\tilde{\mu} = 0$, $\tilde{\beta} = 0$ and $\tilde{\Sigma} = I$ (identity matrix). In contrast, Table 2 shows the results taking into account that the systematic starting point is such that maximize the likelihood function for univariate linear combination $Y = B_j X$ and considering the EM algorithm with parameter free λ. Specifically, our starting point is $\tilde{\lambda} = -0.97$, $\tilde{\gamma} = 1.31$, $\tilde{\mu} = 0, \tilde{\beta} = 0$ y $\tilde{\Sigma} = I$ (identity matrix). Note that the value of log-likelihood is greater when we do not estimate with λ fixed, the p-value of the empirical test of goodness of fit is higher and the number of iterations is smaller.

Table 2.

Estimation via the EM algorithm with free λ and systematic starting point					
Iterations	353				
p-value	0.437				
Log-likelihood	79952.42				
λ	-0.327				
δ	0.005				
γ	264.299				
Matrix Δ					
Currency	CHF	DEM	GBP	CAD	JPY
CHF	2.72143	2.14245	1.42368	0.00846	1.32138
DEM	2.14245	2.16606	1.28088	0.03283	1.12272
GBP	1.42368	1.28088	1.86350	0.03869	0.77683
CAD	0.00846	0.03283	0.03869	0.43249	0.00932
JPY	1.32138	1.12272	0.77683	0.00932	2.33139
Currency	CHF	DEM	GBP	CAD	JPY
β	22.4104	-15.9031	-13.7836	-10.0267	12.5831
μ	-0.0005	-0.0002	0.0001	0.0001	-0.0006

6. Conclusions

This paper shows how the choice of a starting point for systematic implementation of the EM algorithm can reduce the number of iterations in the estimation of the parameters of the probability distribution fitted to the time series of exchange rates: Swiss franc, the Deutschemark, the British pound, the Canadian dollar and the Japanese yen. At the same time parameter adjustment λfree achieves a higher optimal value in log-likelihood function of the GH family.

The results are corroborated by empirical test of multivariate goodness of fit and we observe a higher significant p-value for the multivariate distribution with free versus the corresponding distribution with = -0.5. Calibration adjustment for GH family is better when the parameter is free and systematic choice of an initial point.

References

[1] Abramowitz, M., & Stegun, I. A. (1972). *Handbook of Mathematical Functions*. New York, USA: Dover; Chapter 9 Bessel Functions of Integer Order.

[2] Barndorff-Nielsen, O. E. (1977). Exponentially decreasing distributions for the logarithm of the particle size. *Proc. R. Soc. A, 353*(1674), 401-419.

[3] Blæsild, P., & Sørensen, M. (1992). *'hyp'- a computer program for analyzing data by means of the hyperbolic distribution*. Research Report 248, Department of Theoretical Statistics, University of Aarhus.

[4] Breymann, W., & Lüthi, D. (2011). *ghyp: A package on generalized hyperbolic distributions*. CRAN, 1-26.

[5] Dagpunar, J. S. (1989). An Easily Implemented Generalized Inverse Gaussian Generator. *Communications in Statistics - Simulation and Computation*, 18(2), 703-710.

[6] Dempster, A. P., Laird, N. M., & Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm (with discussion). *Journal Of The Royal Statistical Society, Series B, 39*(1), 1-38.

[7] Eberlein, E., & Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli, 1*(3), 281-299.

[8] Hu, W. (2005). *Calibration of multivariate generalized hyperbolic distributions using the EM algoritm, with applications in risk management, portfolio optimization and portfolio credit risk*. Electronic Theses, Treatises and Dissertations. Paper 3694. The Florida State University.

[9] Justel, A., Peña, D., & Zamar, R. (1997). A multivariate Kolmogorov-Smirnov test of goodness of fit. *Statistics & Probability Letters*, *35*(3), 251-259.

[10] Gábor, J. S., & Rizzo, M. L. (2004). *Testing for equal distributions in high dimension.* Working paper. Bowling Green State University and Ohio University.

[11] Kinderman, A. J. (1977). Computer Generation of Random Variables Using the Ratio of Uniform Deviates. *ACM Transactions on Mathematmal Software* , 257-260.

[12] Liu, C., & Rubin, D. B. (1995). ML estimation of the t distribution using EM and its extensions, ECM and ECME. *Statistica Sinica*, *5*(1995), 19-39.

[13] Loudin, J., & Hannu, M. (2003). *A multivariate method por comparing n-dimensional distributios*. Phystat 2003, SLAC, Stanford, California.

[14] Lüthi, D. (2011). *Package ghyp*. CRAN, 1-50.

[15] McAssey, M. P. (2013). An empirical goodness-of-fit test for multivariate distributions. *Journal of Applied Statistics, 40*(5),1120-1131.

[16] McNeil, A. J., Frey, R., & Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press.

[17] Meng, X.-L., & Rubin, D. B. (1993). Maximum likelihood estimation via the ECM algorithm: a general framework. *Biometrika, 80*(2), 267–278.

[18] Paolella, M. S. (2007). *Intermediate Probability*. West Sussex, England: John Wiley & Sons.

[19] Prause, K. (1999). *The Generalized hyperbolic model: estimation, financial derivatives, and risk measures*. Ph.D. Thesis, University of Freiburg.

[20] Protassov, R. S. (2004). EM-based maximum likelihood parameter estimation for multivariate generalized hyperbolic distributions with fixed λ. *Statistics and Computing*, *14*(1), 67-77.

[21] R Development Core Team (2011). *R: A language and environment for statistical computing*. R Foundation for Statistical Computing,Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org/.

[22] Scott, D. (2012). *The Generalized Hyperbolic Distribution*. CRAN , 1-77.

[23] Trejo Becerril, B. R., Lorenzo Valdés, A., & Núñez Mora, J. A. (2006). Distribución de los rendimientos del mercado mexicano accionario. *Estudios Económicos*, *21*(1), 85-118.

