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Portfolio Optimization via Generalized Multivariate Shrinkage

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Abstract

The shrinkage method of Ledoit and Wolf (2003; 2004a; 2004b) has shown certain success in estimating a well-conditioned covariance matrix for high dimensional portfolios. This paper generalizes the shrinkage method of Ledoit and Wolf to a multivariate shrinkage setting, by which the well-conditioned covariance matrix is estimated using the weighted averaging of multiple priors, instead of single ones. In fact, it can be argued that the generalized multivariate shrinkage approach reduces estimation errors and uncertainty when projecting the true covariance matrix onto the line, spanned by priors joining to the sample covariance matrix. Hence, the generalized multivariate shrinkage is less subjected to sampling variation. Empirically, I use the U.S. firms to form portfolios for out-of-sample forecast. Using Ledoit and Wolf's approach as benchmark, out-of-sample portfolios constructed from the proposed method gain significant variance reductions and sizable improvement of information ratios.

JEL Classifications: G11, G12

Keywords: generalized multivariate shrinkage, portfolio selection, covariance estimation, shrinkage intensity, multiple priors

1. Introduction

It is a long-standing difficulty to estimate a well-conditioned and invertible variance-covariance matrix for a high dimensional portfolio selection. To address this issue, Ledoit and Wolf (2003; 2004a; 2004b) optimize investment portfolios by shrinking between a structured estimator and the sample covariance matrix to gain the trade-off between estimation errors and bias. The trade-off between the bias and variance is realized through shrinkage weights in the projection of the true covariance matrix onto the geometric line between a structure estimator and the sample covariance matrix. Their shrinkage method has shown theoretically and empirically attractive to the covariance estimation problem of a high dimensional portfolio in that it guarantees obtaining a well-defined and invertible variance-covariance matrix.

The shrinkage approach is also referred to as the empirical Bayesian shrinkage, e.g., Ledoit and Wolf (2004b) and DeMiguel, Garlappi, Nogales, and Uppal (2009), with a natural Bayesian interpretation for the trade-off. However, in their contexts, the Bayesian decision-maker is assumed to depend on only one single prior (a single structure matrix), or, equivalently, to be neutral to uncertainty of priors in the sense of Knight (1921). Given the difficulty in estimating moments of asset returns, and the sensitivity to the choice of a particular prior, it is important to consider multiple priors and hence desire robust portfolio rules that work well for a set of possible models.

A few works have shown the improvement of the portfolio performance via multiple priors. Garlappi, Uppal, and Wang (2007) develop a model with multiple priors and aversion to ambiguity to examine the normative implications of parameter and model uncertainty for investment management. They have shown their portfolios with multiple priors are more stable over time and deliver a higher out-of-sample Sharpe ratio. Furthermore, their model with multiple priors has several attractive features: (1) it has a solid axiomatic foundation; (2) it is flexible enough to allow for different degrees of uncertainty; (3) it delivers closed-form expressions for the optimal portfolio. The further work on the applications of multiple priors is Lutgens and Schotman (2010) who study robust portfolio optimization with multiple experts. Their work, similar to Black and Litterman (1992), combine the distinct priors of a finite set; nonetheless, the difference is that they attach prior probabilities to each of the priors, by assuming the aversion to ambiguity of a robust investor. They demonstrate that the multi-prior model can be applied to the practical problem of portfolio selection if there are estimation errors.

This paper generalizes the shrinkage method of Ledoit and Wolf to a multivariate shrinkage setting, in which we introduce multiple priors to estimate a well-conditioned covariance matrix, instead of single ones. The key of Ledoit and Wolf method is to determine shrinkage weights between a selected structure matrix and the sample covariance matrix. Nonetheless, the procedure of selecting a structure matrix itself can generate great uncertainty for estimating shrinkage weights. To tackle this issue in the projection of a true covariance matrix, I propose targeting multiple structure covariance matrices through a weighted averaging, by which more information can be absorbed to facilitate the estimation of shrinkage intensities.

As shown in Lutgens and Schotman (2010), dispersed priors might lead to averaging of the alternatives, whereas one prior might dominate others when they agree on the main generating factors. Since a robust portfolio looks at different sets of priors, and for each potential portfolio evaluates the worst case, the robust portfolio is less likely to overestimate the available investment opportunities, especially when the different priors disagree. Accordingly, I conjecture that the robust portfolio with multiple priors can exhibit better out-of-sample performance.

This paper constructs mean-variance and global minimum-variance portfolios with various sizes, and constraints of expected returns in out-of-sample forecast. The results show that compared to the single targeting method, the generalized multivariate shrinkage performs better out-of-sample in terms of significantly reducing portfolio variance across different portfolio types, sizes, and constraints of conditional expected returns, whereas it has higher information ratios merely for small and medium sized portfolios. Additionally, the out-of-sample portfolios with multiple priors appear more attractive to an active portfolio manager in that they always have lower portfolio turnovers than those from single ones. Furthermore, in many cases a multivariate shrinkage outperforms the single shrinkage which is estimated by any one of the components/priors used in that multivariate shrinkage.

This paper is structured as follows. Section 2 reviews the literature. Section 3 derives the optimal generalized multivariate shrinkage intensity. Section 4 provides interpretation for multivariate shrinkage. Section 5 presents the consistent estimator of the optimal generalized multivariate shrinkage intensity. Section 6 constructs portfolios and selects targeting matrices for multivariate shrinkage. Section 7 describes data. Section 8 reports empirical results. Section 9 concludes the paper.

2. Literature Review

Shrinkage estimators have been widely studied to reduce the estimation error on the estimation of the moments of asset returns. DeMiguel, Martin-Utrera, and Nogales (2011) study shrinkage estimators for the vector of means, the covariance matrix and portfolio weights themselves and provide a theoretical and empirical analysis of different new methods to calibrate shrinkage estimators within portfolio optimization. Candelon, Hurlin, & Tokpavi (2012) introduce a new framework based on shrinkage estimators to improve the performance of small size portfolios and illustrate that double shrinkage estimation can be beneficial for realistic small estimation sample sizes. Golosnoy and Okhrin (2009) derive the flexible shrinkage estimator for the optimal portfolio weights, which allows dynamic adjustments of model structure. Their estimator is based on grouping the assets in order to capture non-homogeneity of estimation risk. The assets are assigned to groups using a clustering procedure with the number of groups determined from the data. The flexible shrinkage approach exhibits sound and robust performance compared to the popular portfolio selection alternatives. Golosnoy and Okhrin (2007) propose applying shrinkage directly to the portfolio weights by using the non-stochastic target vector. They estimated the classical Markowitz weights which are shrunk to the deterministic target portfolio. Ledoit and Wolf (2012) extend the shrinkage method by considering nonlinear transformations of the sample eigenvalues, using recent results from Random Matrix Theory (RMT). Behr, Guettler, and Miebs (2012) impose the set of constraints that yields the optimal trade-off between sampling error reduction and bias for the variance-covariance matrix. Their results show that their constrained minimum-variance portfolio yields significantly lower out-of-sample variances than many established minimum-variance portfolio strategies. Kourtis, Dotsis, and Markellos (2012) apply shrinkage method directly to the inverse covariance matrix using two non-parametric methods. Their strategies have an intuitive form which allows for accounting for short-sale constraints, high transaction costs and singular covariance matrices. Their new strategies generally offer higher risk-adjusted returns and lower levels of risk.

The alternative methods to shrinkage approach are also studied and compared in the literature. Jagannathan and Ma (2003) show that constraining portfolio weights to be non-negative is equivalent to using the sample covariance matrix after reducing its large elements and then form the optimal portfolio without any restrictions on portfolio weights. They explain why constraining portfolio weights to be nonnegative can reduce the risk in estimated optimal portfolios even when the constraints are wrong. Surprisingly, with no-short-sale constraints in place, the sample covariance matrix performs as well as covariance matrix estimates based on factor models, shrinkage estimators, and daily data. DeMiguel *et al*. (2009) solve the traditional minimum-variance problem subject to the additional constraint that the norm of the portfolio-weight vector be smaller than a given threshold. They provide a moment-shrinkage interpretation and a Bayesian interpretation where the investor has a prior belief on portfolio weights rather than on moments of asset returns. They find that the norm-constrained portfolios often have a higher Sharpe ratio than the portfolio strategies in Jagannathan and Ma (2003), Ledoit and Wolf (2003; 2004a; 2004b), the 1/N portfolio and other strategies in the literature such as factor portfolios. Liu and Lin (2010) compare portfolio performance among different shrinkage methods and find that taking a simple average of the historical sample covariance matrix and the covariance matrix estimated from the single-index model provides the best overall performance among all competing methods considered in their work. Briner and Connor (2008) find that the factor model performs best for large investment universes and typical sample lengths. Wolf (2006) compares shrinkage estimation to resampled efficiency and find that the resampling method improves upon the sample covariance matrix. But, according to the empirical study, the resampling method is inferior to shrinkage estimation. Resampling can be combined with shrinkage. However, Wolf (2006) cannot find evidence that this combination offers any further improvement beyond ‘pure’ shrinkage.

3. Generalized Multivariate Optimal Shrinkage Intensity

This section derives the solution of multivariate shrinkage model. The multivariate shrinkage approach is a natural extension to the single targeting of Ledoit and Wolf. Following the notation convention, let X denote a $N \times T$ matrix. T denotes number of observations on a system of N random variables representing T returns on a universe of N stocks.

Let F be a $K \times (N \times N)$ matrix such that $F = (f^1, \dots, f^K)'$, where f^k is a $N \times N$ matrix for $k = 1, \dots, K$. Hence, F contains K priors in the given system. Theoretically, K can be infinity; however, it is practically inefficient.

Consider the optimization problem:

$$\min_{\alpha} E\left[\left\|\Sigma^* - \Sigma\right\|^2\right]$$

$$s.t.\ \Sigma^* = (\alpha \otimes I)' F + (1 - \alpha' \mathbf{1}) S \tag{1}$$

where $\|\cdot\|^2$ is Frobenius norm. α is a $K \times 1$ vector of weights, $(\alpha_1, \dots, \alpha_K)'$ with $\alpha_k \in [0,1]$ and $\sum_{k=1}^{K} \alpha_k \in [0,1]$. Equation (1) has that $(\alpha \otimes I)' F = \sum_{k=1}^{K} \alpha_k f^k$ with $\otimes$ being the Kronecker product. I is an $N \times N$ identity matrix and **1** denotes a conformable vector of ones. Σ^*, depending on α, is the linear convex combination of the multivariate shrinkage targeting matrices , F, and the sample covariance matrix, S. Σ is the population covariance matrix with elements σ_{ij} *for* $i, j = 1, \dots, N$.

Hence, the quadratic loss function can be written as follows

$$L(\alpha) = \left\|(\alpha \otimes I)' F + (1 - \alpha' \mathbf{1}) S - \Sigma\right\|^2 \tag{2}$$

The corresponding risk function is:

$$R(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha' var(f_{ij} - \mathbf{1}s_{ij})\alpha + var(s_{ij}) + 2\alpha' cov(f_{ij} - \mathbf{1}s_{ij}, s_{ij}) + \left[\alpha'(\phi_{ij} - \mathbf{1}\sigma_{ij})\right]^2 \tag{3}$$

where f_{ij} is a $K \times 1$ vector with the elements $(f_{ij}^1, \dots, f_{ij}^K)$. f_{ij}^k is the element of the i^{th} row and j^{th} column of the $N \times N$ matrix f^k, $k = 1, \dots, K$. Further, $E(s_{ij}) = \sigma_{ij}$ is a scalar , and $E(f_{ij}) = \phi_{ij}$ is a $K \times 1$ vector. Note that

$$cov(f_{ij} - \mathbf{1}s_{ij}, s_{ij}) = \begin{pmatrix} cov(f_{ij}^1, s_{ij}) - var(s_{ij}) \\ \vdots \\ cov(f_{ij}^K, s_{ij}) - var(s_{ij}) \end{pmatrix} \begin{pmatrix} cov(f_{ij}^1, s_{ij}) - var(s_{ij}) \\ \vdots \\ cov(f_{ij}^K, s_{ij}) - var(s_{ij}) \end{pmatrix}$$

is a $K \times 1$ vector. Minimize the risk function, $R(\alpha)$, with respect to α to get:

$$F.O.C.: \quad \frac{\partial R(\alpha)}{\partial \alpha} = 2\sum_{i=1}^{n}\sum_{j=1}^{n} var(f_{ij} - \mathbf{1}s_{ij})\alpha + cov(f_{ij} - \mathbf{1}s_{ij}, s_{ij}) + (\phi_{ij} - \mathbf{1}\sigma_{ij})(\phi_{ij} - \mathbf{1}\sigma_{ij})'\alpha \tag{4}$$

$$S.O.C.: \quad \frac{\partial^2 R(\alpha)}{\partial \alpha^2} = 2\sum_{i=1}^{n}\sum_{j=1}^{n} var(f_{ij} - \mathbf{1}s_{ij}) + (\phi_{ij} - \mathbf{1}\sigma_{ij})(\phi_{ij} - \mathbf{1}\sigma_{ij})' \tag{5}$$

Note that equation (5) is a positive definite matrix such that there exists a solution for minimizing equations (1) and (3).

Setting the first order condition to be zero and solve for the optimal weight vector α^*, I derive the solution for the optimal multivariate shrinkage intensity as

$$\alpha^* = \left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\left[var(f_{ij} - \mathbf{1}s_{ij}) + (\phi_{ij} - \mathbf{1}\sigma_{ij})(\phi_{ij} - \mathbf{1}\sigma_{ij})'\right]\right\}^{-1} \left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\left[var(s_{ij})\mathbf{1} - cov(f_{ij}, \mathbf{1}s_{ij})\right]\right\} \tag{6}$$

where

$$(\phi_{ij} - \mathbf{1}\sigma_{ij})(\phi_{ij} - \mathbf{1}\sigma_{ij})' = \begin{pmatrix} (\phi_{ij}^1 - \sigma_{ij})^2 & \cdots & (\phi_{ij}^1 - \sigma_{ij})(\phi_{ij}^K - \sigma_{ij}) \\ \vdots & \ddots & \vdots \\ (\phi_{ij}^K - \sigma_{ij})(\phi_{ij}^1 - \sigma_{ij}) & \cdots & (\phi_{ij}^K - \sigma_{ij})^2 \end{pmatrix}$$

is a $K \times K$ matrix. See *Appendix A* for the proof of the optimal multivariate shrinkage intensity of equation (6).

Remark 1: The main difference of equation (6) from Ledoit and Wolf is now that α^* is a vector solution for multivariate targeting matrices. The asymptotic theorem and properties derived in Ledoit and Wolf (2003, 2004a&b) are still valid in equation (6).

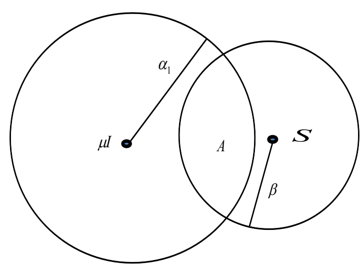
4. Interpretation

An interpretation for the generalized multivariate shrinkage from frequentist statistics is based on the properties of conditional variances and covariances:

$$E[\Sigma \mid \Xi \subset F] \geq E[\Sigma \mid F]$$

where Ξ is a subset of F. That is, one never does worse for predicting Σ when additional information are conditioned on.

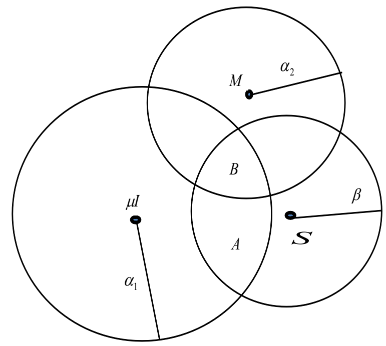
To understand the intuition underlying the multi-prior model, I provide a Bayesian interpretation for the generalized multivariate shrinkage. Σ^* in equation (1) can be seen as the combination of two signals: prior information and sample information. Prior information states that the true covariance matrix Σ lies on the sphere centered around the shrinkage target μI with radius α_1.[1] Sample information states that Σ lies on another sphere, centered around the sample covariance matrix S with radius β. Bringing together prior and sample information, Σ must lie on the intersection of the two spheres, which is the area A in Graph 1. At the center of the area A stands Σ^*.



Graph 1

However, to address sensitivity and uncertainty to a prior, I introduce the second prior information, saying M, which also states that the true covariance matrix Σ lies on the sphere centered around the shrinkage target M with radius α_2. Now, bringing together the two priors and sample information, Σ must lie on the intersection of these three spheres, which is the area B in Graph 2. At the center of area B stands Σ^*. It can be seen that the area B is smaller than the area A in Graph 1. This uncertainty reduction improves the precision to locate Σ^* by reducing variance.

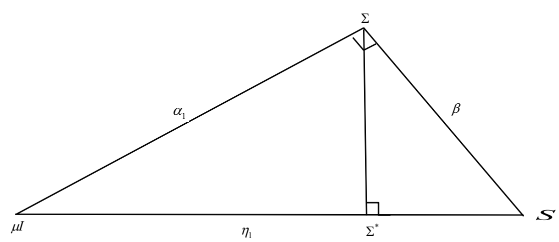
[1] See the definitions of α_1, α_2, α_3, η_1, η_2, η_3, β in *Appendix B*



Graph 2

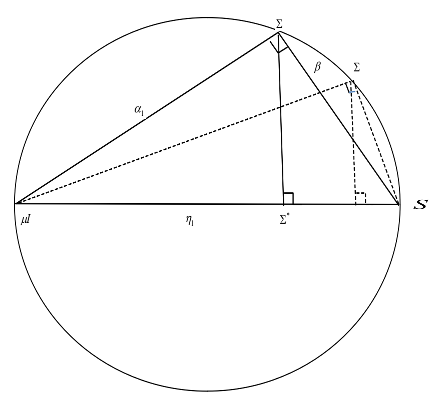
Theoretically, the number of structure targeting matrices can be infinite at the computation cost. Notice that Ledoit and Wolf (2004b) choose the shrinkage targeting, μI since μI is all bias and no variance. By contrast, the sample matrix, S, is consistent but has a lot of estimation errors. Similarly, other structure targeting matrices, e.g., market model and constant correlation matrix, are bias but much less variation.

An alternative way to interpret the multivariate shrinkage philosophy is geometric. Lemma 2.1 in Ledoit and Wolf (2004b) is a projection theorem in Hilbert space, a rewriting of the Pythagorean Theorem as in Graph 3. Σ^* is the projection of Σ on the line between μI and S.



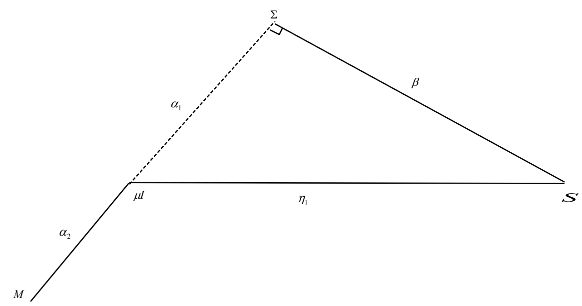
Graph 3

However, since Σ is unknown, Graph 3, merely based on the single shrinkage targeting, μI, cannot uniquely determine the true Σ as showed in Graph 4. The variance for Σ estimation is a circle with radius $\frac{\eta_1}{2}$. Accordingly, the projection range for Σ^* is the line between μI and S.



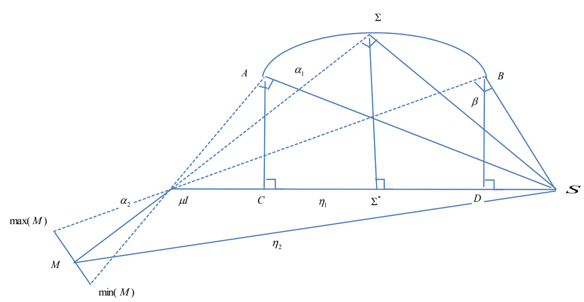
Graph 4

To reduce the variance for Σ or the projection range for Σ^*, I introduce the other shrinkage targeting, say M. M targeting alone by replacing μI has the analog properties with S and Σ as in Graph 3. However, when combine M, μI and S together, I first determine a line between M and μI. Then, based on the line between M and μI, I can uniquely determine a point for Σ, given that M has no estimation errors, as illustrated in Graph 5.



Graph 5

However, in general, the structural targeting matrix, M, still exists to some extent variation as illustrated in Graph 6, while its variation is much less than the sample matrix, S. This provides a variance reduction in Σ or a projection range reduction for Σ^*, which is much smaller than that in Graph 4. Now the Σ^* variation for Σ is no longer a circle but reduced to an arc-shaped radiant, $\widehat{AB}$ as in Graph 6, whose projection for Σ^* is only a partition (CD) of the line between μI and S.



Graph 6

5. Consistent Estimator of Optimal Multivariate Shrinkage Intensity

The rest of this paper takes the optimal multivariate shrinkage intensity of equation (6) for estimation. As shown by Ledoit and Wolf (2003), the optimal shrinkage intensity vanishes asymptotically of the order $O(1/T)$; thus, for simplifying the estimation, the optimal multivariate shrinkage intensity takes the form as

$$\alpha^* = \frac{1}{T}\Gamma^{-1}(\Pi - \Theta) + O(\frac{1}{T^2}) \tag{7}$$

Γ, a $K \times K$ matrix, is an asymptotic estimator for the first term of the right hand side of equation (6), while the rest of terms in equation (6) are asymptotically estimated by Π and Θ, both $K \times 1$ vectors. The equation (7) can be verified by Theorem 1 of Ledoit and Wolf (2003).

The constant $K \times 1$ vector, $\Lambda = \Gamma^{-1}(\Pi - \Theta)$ with elements λ_k, is estimated asymptotically. With Lemma 1&2 of Ledoit and Wolf (2003) and the formula of shrinkage intensity in *Appendix B* of

Ledoit and Wolf (2004a), I can asymptotically estimate each element of Π and Θ. The element estimation for Π is given by $\sum_{i=1}^{N}\sum_{j=1}^{N}\pi_{ij}$, where $\pi_{ij} = AsyVar[\sqrt{T}s_{ij}]$. Note that π_{ij} is the same for different prior matrices. A consistent estimator for π_{ij} is $\hat{\pi}_{ij} = \frac{1}{T}\sum_{t=1}^{T}\{(x_{it}-\bar{x}_i)(x_{jt}-\bar{x}_j)-s_{ij}\}^2$

The element estimation for Θ is given by $(\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}^{k})_{k=1,\ldots,K}$, where $\rho_{ij}^{k} = AsyCov[\sqrt{T}f_{ij}^{k}, \sqrt{T}s_{ij}]$. In the empirical section, I consider four shrinkage targeting matrices, namely market model, constant correlation, identity and diagonal shrinkage. A consistent estimator for ρ_{ij}^{k} of market model is given by

$$\hat{\rho}_{ij}^{k} = \frac{s_{j0}s_{00}(x_{it}-\bar{x}_i)+s_{i0}s_{00}(x_{jt}-\bar{x}_j)-s_{i0}s_{j0}(x_{0t}-\bar{x}_0)}{s_{00}^2}(x_{0t}-\bar{x}_0)(x_{it}-\bar{x}_i)(x_{jt}-\bar{x}_j)-f_{ij}^{k}s_{ij}$$

where x_{0t} and $\bar{x}_0$ is the market return and its sample mean, s_{00} is the variance of the market return, s_{i0} is the covariance between stock i and the market return, s_{ij} is the covariance between stock i and j.

A consistent estimator for ρ_{ij}^{k} of constant correlation is given by

$$\rho_{ij}^{k} = AsyVar\left[\sqrt{T}s_{ii}\right]+\sum_{j=1,j\neq i}^{N}AsyCov\left[\sqrt{T}\bar{r}\sqrt{s_{ii}s_{jj}}, \sqrt{T}s_{ij}\right]$$

where $\bar{r} = \frac{2}{(N-1)N}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}r_{ij}$ for $r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}}$.

From Ledoit and Wolf (2003), elements on the diagonal are

$$AsyVar\left[\sqrt{T}s_{ii}\right] = \hat{\pi}_{ii} = \frac{1}{T}\sum_{t=1}^{T}\{(x_{it}-\bar{x}_i)^2 - s_{ii}\}^2$$

By use of the delta method, $AsyCov\left[\sqrt{T}\bar{r}\sqrt{s_{ii}s_{jj}}, \sqrt{T}s_{ij}\right]$ can be consistently estimated by

$$\frac{\bar{r}}{2}\left(\sqrt{\frac{s_{jj}}{s_{ii}}}AsyCov\left[\sqrt{T}s_{ii}, \sqrt{s_{ij}}\right]+\sqrt{\frac{s_{ii}}{s_{jj}}}AsyCov\left[\sqrt{T}s_{jj}, \sqrt{T}s_{ij}\right]\right)$$

Standard theory implies that a consistent estimator for $AsyCov\left[\sqrt{T}s_{ii}, \sqrt{s_{ij}}\right]$ is given by

$$\frac{1}{T}\sum_{t=1}^{T}\{(x_{it}-\bar{x}_i)^2 - s_{ii}\}\{(x_{it}-\bar{x}_i)(x_{jt}-\bar{x}_j)-s_{ij}\}$$

For indentity and diagonal models, I have $\rho_{ij}^{k} = 0$ and $\rho_{ij}^{k} = \sum_{i=1}^{N}\pi_{ii}$, respectively.

Applying Theorem 1 and Lemma 3 of Ledoit and Wolf (2003), I have $\sum_{i=1}^{N}\sum_{j=1}^{N}var(f_{ij}-\mathbf{1}s_{ij}) = O(1/T)$ such that $\Gamma = (\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij})$, where $\gamma_{ij} = (\phi_{ij}-\sigma_{ij})(\phi_{ij}-\sigma_{ij})'$. A consistent estimator for γ_{ij} is given by its sample counterpart $\hat{\gamma}_{ij} = (f_{ij}-s_{ij})(f_{ij}-s_{ij})'$.

Finally, the optimal multivariate shrinkage intensity is determined by:

$$\alpha_k^* = min(1, \lambda_k/T)$$

In the case of $\sum_{k=1}^{K}\alpha_k^* > 1$, it is rescaled by $\tilde{\alpha}_k = \alpha_k^*/\sum_{k=1}^{K}\alpha_k^*$ as the optimal shrinkage intensity.

6. Portfolios and Multivariate Shrinkage Targeting Matrices

Consider a general mean-variance portfolio (MVP) of Markowitz (1952) type with a universe of N stocks, whose returns are distributed with mean vector μ, and covariance matrix, Σ. Markowitz (1952) defines the problem of portfolio selection as:

$$\min_{w} \; w'\Sigma w$$

$$s.t.\; w'\mathbf{1} = 1$$

$$w'\mu = q$$

where $\mathbf{1}$ denotes a conformable vector of ones, and q is the expected rate of return that is required on the portfolio as a constraint. The well-known solution is

$$w_{MVP} = \frac{C - qB}{AC - B^2}\Sigma^{-1}\mathbf{1} + \frac{qA - B}{AC - B^2}\Sigma^{-1}\mu \tag{8}$$

with $A = \mathbf{1}'\Sigma^{-1}\mathbf{1}$, $B = \mathbf{1}'\Sigma^{-1}\mu$ and $C = \mu'\Sigma^{-1}\mu$.

In this paper, I also estimate a global minimum variance portfolio (GMVP) as

$$\min_{w} \; w'\Sigma w$$

$$s.t.\; w'\mathbf{1} = 1$$

with its solution as

$$w_{GMVP} = \Sigma^{-1}\mathbf{1}\left(\mathbf{1}'\Sigma^{-1}\mathbf{1}\right)^{-1} \tag{9}$$

Note that the solutions of both equations (8) and (9) involve the inverse of the covariance matrix. The conventional approach is to use the sample covariance matrix, $\hat{\Sigma}$ to approximate the population matrix, Σ. However, in a high dimensional portfolio selection problem, the sample covariance is typically not well-conditioned and may not even be invertible. The generalized multivariate shrinkage method obtains an estimator that is both well-conditioned and more accurate than the sample covariance matrix asymptotically. The estimator is distribution-free and has a simple explicit formula that is easy to compute and interpret.

The shrinkage estimator is expressed as a weighted average of the multivariate shrinkage targeting matrices and the sample covariance matrix as

$$\Sigma^* = \alpha' F + (1 - \alpha'\mathbf{1})\hat{\Sigma} \tag{10}$$

where α is a shrinkage intensity vector with $\alpha_k \in [0,1]$, and $\sum_{k=1}^{K}\alpha_k \in [0,1]$. The shrinkage intensity reflects the trade-off of estimation errors and bias. Σ^* as the estimator of Σ is used in equations (8) and (9) to obtain optimal portfolio weights.

To determine the components of F, I consider four types of targeting matrices which have been applied in the literature, namely identity (I), market (M), constant correlation (C), and diagonal (D) targeting matrices, all of which have been applied in Ledoit and Wolf (2003; 2004a; 2004b), DeMiguel *et al*. (2009) and Disatnik and Benninga (2007). S&P500 is used as market factor. Besides four single targeting matrix estimations, I estimate portfolio based on 11 different combinations of these targeting matrices: $\{IM, IC, ID, MC, MD, CD, IMC, IMD, ICD, MCD, IMCD\}$. Each of the combinations is a multivariate shrinkage with the priors more than one. For instance, MC represents that the multivariate shrinkage consists of two targeting matrices, $F = (M, C)'$. In empirical section, I compare out-of-sample performance of multivariate shrinkage to single targeting shrinkage of Ledoit and Wolf.In particular, the formula for shrinkage targeting matrices are as follows.

$$I = \frac{1}{N}\sum_{i=1}^{N} s_{ii} I_N \quad D = diag(\hat{\Sigma})$$

where I_N is a $N \times N$ identity matrix and $diag(\cdot)$ is a diagonal matrix.

Let c_{ij} be the elements of the constant correlation targeting matrix. I have C estimated as $c_{ii} = s_{ii}$ and $c_{ij} = \bar{r}\sqrt{s_{ii}s_{jj}}$. Further, for market model, I obtain the market risk beta for individual stocks by the following regression

$$x_{it} = \alpha_i + \beta_i x_{0t} + \varepsilon_{it}$$

where residuals ε_{it} are uncorrelated to market return x_{0t}. Let δ_{ii} denote the variance of residual ε_{it} and s_{00} for the market return sample variance. The covariance matrix implied by this market model is

$$M = s_{00}^2 \beta\beta' + \Delta$$

where β is the $N \times 1$ vector containing the estimated risk beta of individual stocks and Δ is the diagonal matrix containing residual variances δ_{ii}.

7. Data

Monthly U.S. stock returns, from January 1980 to December 2010, were taken from the Center for Research in Security Prices (CRSP). The portfolios with monthly rebalancing are constructed similar to DeMiguel *et al*. (2009) and Jagannathan and Ma (2003): in April of each year I randomly select N assets among all assets in the CRSP data set for which there is return data for the previous 120 months as well as for the next 12 months. I then consider these randomly selected N assets as the asset universe for the next 12 months.

Kirby and Ostdiek (2012) have shown that targeting conditional expected excess returns (q determined by investors in equation (8)) have greatly affected out-of-sample performance of portfolios; thus, in this paper, the different expected return constraints are considered for $q = (3\%, 8\%, 12\%, 16\%, 20\%)$ annually. I also estimate for different portfolio sizes, $N = (30, 50, 80, 100, 225, 500)$. This range of portfolio sizes covers the important benchmarks as DJIA, Xetra DAX, DJ STOXX 50, FTSE 100, NASDAQ-100, NIKKEI 225, and S&P 500, similar to Ledoit and Wolf (2004a). I take the average of realized returns over the estimation window of the past 120 months for expected returns, μ in equation (8).

8. Empirical Results

The generalized multivariate shrinkage performance is compared to the single targeting performance of Ledoit and Wolf approach in terms of: (i) reduction in out-of-sample portfolio standard deviation; (ii) improvement in out-of-sample portfolio information ratio; and (iii) lower portfolio turnover. The turnover is defined as the total turnover of Grinold and Kahn (2000, Chapter 16) and DeMiguel *et al*. (2009). In general, the higher the turnover is, the less attractive the portfolio is to an active manager.

Table 1: Standard Deviation and Robust Tests of Out-of-Sample Portfolio Forecasting

	[a]: 300bp						[b]: 800bp						[c]: 1200bp					
	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500
Lediot and Wolf																		
I	0.5487	0.4390	0.3745	0.3683	0.3288	0.3109	0.4508	0.3834	0.3519	0.3439	0.3132	0.3065	0.4160	0.3736	0.3576	0.3487	0.3190	0.3152
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
M	0.5069	0.4036	0.3404	0.3329	0.2750	0.2457	0.4205	0.3562	0.3159	0.3058	0.2597	0.2362	0.3927	0.3515	0.3173	0.3054	0.2642	0.2410
	(0.008)	(0.002)	(0.003)	(0.004)	(0.005)	(0.003)	(0.000)	(0.000)	(0.009)	(0.001)	(0.008)	(0.004)	(0.008)	(0.003)	(0.002)	(0.009)	(0.001)	(0.000)
C	0.4931	0.3934	0.3328	0.3257	0.2702	0.2422	0.4150	0.3585	0.3237	0.3183	0.2768	0.2485	0.3972	0.3656	0.3397	0.3369	0.3011	0.2674
	(0.003)	(0.005)	(0.000)	(0.007)	(0.002)	(0.001)	(0.002)	(0.009)	(0.009)	(0.004)	(0.007)	(0.003)	(0.009)	(0.009)	(0.004)	(0.007)	(0.000)	(0.002)
D	0.5233	0.4090	0.3467	0.3361	0.2756	0.2487	0.4326	0.3587	0.3242	0.3106	0.2603	0.2393	0.4002	0.3531	0.3281	0.3124	0.2657	0.2443
	(0.008)	(0.004)	(0.005)	(0.001)	(0.005)	(0.005)	(0.009)	(0.009)	(0.008)	(0.000)	(0.002)	(0.005)	(0.004)	(0.047)	(0.000)	(0.040)	(0.018)	(0.006)
(1) Average	0.5180	0.4112	0.3486	0.3408	0.2874	0.2619	0.4297	0.3642	0.3289	0.3196	0.2775	0.2576	0.4015	0.3609	0.3357	0.3258	0.2875	0.2670
Generalized Multivariate Shrinkage																		
IM	0.5204	0.4155	0.3508	0.3435	0.2867	0.2562	0.4288	0.3658	0.3245	0.3152	0.2706	0.2471	0.3987	0.3590	0.3248	0.3146	0.2746	0.2522
	(0.004)	(0.009)	(0.009)	(0.007)	(0.006)	(0.007)	(0.003)	(0.005)	(0.003)	(0.001)	(0.005)	(0.003)	(0.007)	(0.002)	(0.008)	(0.006)	(0.005)	(0.008)
IC	0.5014	0.4052	0.3433	0.3373	0.2850	0.2588	0.4159	0.3645	0.3294	0.3261	0.2899	0.2670	0.3947	0.3666	0.3419	0.3422	0.3126	0.2868
	(0.008)	(0.004)	(0.002)	(0.003)	(0.001)	(0.003)	(0.001)	(0.008)	(0.008)	(0.006)	(0.001)	(0.001)	(0.004)	(0.888)	(0.003)	(0.002)	(0.197)	(0.000)
ID	0.5384	0.4294	0.3642	0.3571	0.3071	0.2886	0.4425	0.3738	0.3401	0.3310	0.2918	0.2836	0.4089	0.3643	0.3440	0.3339	0.2971	0.2916
	(0.000)	(0.003)	(0.001)	(0.007)	(0.000)	(0.005)	(0.003)	(0.001)	(0.009)	(0.006)	(0.017)	(0.006)	(0.009)	(0.091)	(0.008)	(0.031)	(0.808)	(0.002)
MC	0.4907	0.3937	0.3329	0.3262	0.2728	0.2394	0.4103	0.3554	0.3218	0.3156	0.2768	0.2454	0.3908	0.3594	0.3363	0.3318	0.2990	0.2643
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
MD	0.5065	0.4036	0.3404	0.3324	0.2747	0.2456	0.4195	0.3559	0.3155	0.3050	0.2591	0.2360	0.3910	0.3508	0.3168	0.3043	0.2636	0.2407
	(0.031)	(0.005)	(0.007)	(0.004)	(0.001)	(0.001)	(0.003)	(0.008)	(0.510)	(0.005)	(0.004)	(0.000)	(0.007)	(0.001)	(0.074)	(0.030)	(0.000)	(0.007)
CD	0.4912	0.3929	0.3319	0.3249	0.2689	0.2401	0.4112	0.3560	0.3211	0.3153	0.2741	0.2459	0.3918	0.3611	0.3360	0.3325	0.2974	0.2647
	(0.031)	(0.005)	(0.007)	(0.004)	(0.001)	(0.001)	(0.003)	(0.008)	(0.510)	(0.005)	(0.004)	(0.000)	(0.007)	(0.001)	(0.074)	(0.030)	(0.000)	(0.007)
(2) Average	0.5081	0.4067	0.3439	0.3369	0.2825	0.2548	0.4213	0.3619	0.3254	0.3180	0.2770	0.2542	0.3960	0.3602	0.3333	0.3266	0.2907	0.2667
IMC	0.4993	0.4018	0.3399	0.3348	0.2787	0.2458	0.4136	0.3601	0.3250	0.3225	0.2828	0.2528	0.3922	0.3617	0.3371	0.3379	0.3050	0.2723
	(0.000)	(0.004)	(0.895)	(0.081)	(0.090)	(0.046)	(0.001)	(0.645)	(0.275)	(0.032)	(0.015)	(0.051)	(0.032)	(0.007)	(0.080)	(0.057)	(0.009)	(0.002)
IMD	0.5239	0.4223	0.3586	0.3542	0.3003	0.2700	0.4309	0.3714	0.3276	0.3209	0.2812	0.2583	0.4010	0.3636	0.3250	0.3171	0.2837	0.2621
	(0.082)	(0.001)	(0.006)	(0.000)	(0.874)	(0.052)	(0.007)	(0.007)	(0.044)	(0.004)	(0.048)	(0.836)	(0.024)	(0.002)	(0.000)	(0.016)	(0.002)	(0.038)
ICD	0.5054	0.4097	0.3468	0.3407	0.2888	0.2627	0.4190	0.3676	0.3295	0.3266	0.2929	0.2708	0.3980	0.3691	0.3404	0.3411	0.3154	0.2910
	(0.005)	(0.006)	(0.002)	(0.005)	(0.721)	(0.014)	(0.059)	(0.013)	(0.003)	(0.008)	(0.731)	(0.285)	(0.011)	(0.020)	(0.024)	(0.001)	(0.002)	(0.977)
MCD	0.4876	0.3927	0.3322	0.3263	0.2719	0.2393	0.4065	0.3537	0.3194	0.3149	0.2755	0.2449	0.3869	0.3575	0.3333	0.3309	0.2976	0.2636
	(0.002)	(0.003)	(0.034)	(0.097)	(0.006)	(0.008)	(0.004)	(0.040)	(0.047)	(0.016)	(0.002)	(0.000)	(0.008)	(0.020)	(0.001)	(0.006)	(0.009)	(0.007)
(3) Average	0.5040	0.4066	0.3444	0.3390	0.2849	0.2544	0.4175	0.3632	0.3254	0.3212	0.2831	0.2567	0.3945	0.3630	0.3339	0.3317	0.3004	0.2723
IMCD	0.5030	0.4085	0.3488	0.3462	0.3010	0.2683	0.4147	0.3646	0.3287	0.3283	0.3004	0.2725	0.3933	0.3650	0.3378	0.3402	0.3195	0.2903
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
(4) Average	0.5062	0.4069	0.3445	0.3385	0.2851	0.2559	0.4193	0.3626	0.3257	0.3201	0.2814	0.2568	0.3952	0.3616	0.3339	0.3297	0.2969	0.2709

continued to Table 1	[d]: 1600bp						[e]: 2000bp						[f]: GMV					
	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500
Lediot and Wolf																		
I	0.4309	0.3975	0.3838	0.3745	0.3401	0.3340	0.4910	0.4498	0.4266	0.4174	0.3741	0.3613	0.4030	0.3673	0.3503	0.3428	0.3127	0.3072
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
M	0.4105	0.3780	0.3375	0.3242	0.2831	0.2560	0.4686	0.4299	0.3733	0.3592	0.3138	0.2798	0.3805	0.3412	0.3107	0.3007	0.2588	0.2361
	(0.001)	(0.001)	(0.001)	(0.008)	(0.005)	(0.007)	(0.003)	(0.001)	(0.009)	(0.009)	(0.003)	(0.008)	(0.005)	(0.007)	(0.007)	(0.009)	(0.008)	(0.001)
C	0.4253	0.4027	0.3738	0.3741	0.3384	0.2961	0.4915	0.4625	0.4216	0.4249	0.3851	0.3322	0.3871	0.3498	0.3195	0.3136	0.2704	0.2438
	(0.008)	(0.009)	(0.004)	(0.009)	(0.007)	(0.005)	(0.007)	(0.000)	(0.007)	(0.007)	(0.005)	(0.002)	(0.006)	(0.008)	(0.002)	(0.009)	(0.005)	(0.009)
D	0.4135	0.3802	0.3509	0.3337	0.2860	0.2598	0.4686	0.4339	0.3893	0.3713	0.3184	0.2842	0.3873	0.3463	0.3210	0.3081	0.2591	0.2382
	(0.007)	(0.007)	(0.008)	(0.024)	(0.001)	(0.007)	(0.016)	(0.008)	(0.037)	(0.001)	(0.004)	(0.041)	(0.008)	(0.001)	(0.007)	(0.003)	(0.002)	(0.009)
(1) Average	0.4200	0.3896	0.3615	0.3516	0.3119	0.2865	0.4799	0.4440	0.4027	0.3932	0.3478	0.3144	0.3895	0.3512	0.3254	0.3163	0.2752	0.2563
Generalized Multivariate Shrinkage																		
IM	0.4169	0.3837	0.3442	0.3339	0.2932	0.2678	0.4779	0.4345	0.3798	0.3699	0.3238	0.2922	0.3852	0.3497	0.3184	0.3096	0.2693	0.2468
	(0.000)	(0.791)	(0.005)	(0.149)	(0.056)	(0.004)	(0.595)	(0.020)	(0.002)	(0.081)	(0.003)	(0.007)	(0.000)	(0.000)	(0.001)	(0.003)	(0.004)	(0.003)
IC	0.4233	0.3995	0.3734	0.3778	0.3486	0.3159	0.4932	0.4566	0.4195	0.4280	0.3943	0.3521	0.3853	0.3549	0.3255	0.3233	0.2854	0.2635
	(0.008)	(0.044)	(0.002)	(0.003)	(0.004)	(0.001)	(0.001)	(0.006)	(0.063)	(0.008)	(0.004)	(0.055)	(0.009)	(0.000)	(0.004)	(0.000)	(0.002)	(0.000)
ID	0.4243	0.3886	0.3680	0.3577	0.3174	0.3095	0.4842	0.4413	0.4086	0.3986	0.3499	0.3357	0.3953	0.3575	0.3371	0.3286	0.2916	0.2839
	(0.015)	(0.002)	(0.009)	(0.032)	(0.003)	(0.002)	(0.008)	(0.048)	(0.003)	(0.008)	(0.003)	(0.005)	(0.002)	(0.006)	(0.009)	(0.008)	(0.007)	(0.000)
MC	0.4182	0.3937	0.3693	0.3670	0.3347	0.2934	0.4844	0.4514	0.4163	0.4165	0.3801	0.3298	0.3817	0.3438	0.3189	0.3130	0.2721	0.2425
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
MD	0.4083	0.3769	0.3369	0.3230	0.2826	0.2559	0.4663	0.4285	0.3728	0.3580	0.3134	0.2798	0.3784	0.3407	0.3100	0.2995	0.2580	0.2356
	(n=30)	(n=50)	(n=80)	(n=100)	(n=225)	(n=500)	(n=30)	(n=50)	(n=80)	(n=100)	(n=225)	(n=500)	(n=30)	(n=50)	(n=80)	(n=100)	(n=225)	(n=500)
CD	0.4188	0.3962	0.3691	0.3688	0.3342	0.2934	0.4844	0.4545	0.4162	0.4193	0.3804	0.3295	0.3816	0.3465	0.3172	0.3116	0.2682	0.2418
	(0.026)	(0.004)	(0.009)	(0.034)	(0.009)	(0.005)	(0.061)	(0.002)	(0.004)	(0.008)	(0.001)	(0.004)	(0.006)	(0.008)	(0.023)	(0.011)	(0.007)	(0.000)
(2) Average	0.4183	0.3898	0.3601	0.3547	0.3184	0.2893	0.4817	0.4445	0.4022	0.3984	0.3570	0.3198	0.3846	0.3489	0.3212	0.3143	0.2741	0.2523
IMC	0.4205	0.3943	0.3686	0.3729	0.3406	0.3017	0.4899	0.4513	0.4150	0.4228	0.3860	0.3383	0.3827	0.3496	0.3219	0.3206	0.2783	0.2499
	(0.075)	(0.075)	(0.074)	(0.578)	(0.024)	(0.001)	(0.092)	(0.318)	(0.000)	(0.059)	(0.009)	(0.047)	(0.016)	(0.871)	(0.006)	(0.023)	(0.004)	(0.078)
IMD	0.4208	0.3876	0.3429	0.3344	0.3016	0.2772	0.4841	0.4381	0.3783	0.3701	0.3325	0.3018	0.3878	0.3538	0.3199	0.3136	0.2799	0.2584
	(0.001)	(0.027)	(0.056)	(0.009)	(0.002)	(0.092)	(0.003)	(0.009)	(0.011)	(0.016)	(0.027)	(0.055)	(0.002)	(0.006)	(0.963)	(0.022)	(0.032)	(0.004)
ICD	0.4276	0.4020	0.3714	0.3761	0.3517	0.3208	0.4988	0.4596	0.4181	0.4265	0.3980	0.3580	0.3904	0.3572	0.3255	0.3229	0.2888	0.2677
	(0.018)	(0.008)	(0.047)	(0.020)	(0.006)	(0.559)	(0.009)	(0.007)	(0.852)	(0.065)	(0.086)	(0.096)	(0.009)	(0.002)	(0.078)	(0.207)	(0.003)	(0.017)
MCD	0.4146	0.3918	0.3661	0.3664	0.3333	0.2925	0.4815	0.4499	0.4133	0.4163	0.3789	0.3288	0.3771	0.3427	0.3160	0.3120	0.2705	0.2417
	(0.099)	(0.032)	(0.002)	(0.000)	(0.005)	(0.000)	(0.055)	(0.929)	(0.004)	(0.004)	(0.004)	(0.004)	(0.008)	(0.007)	(0.008)	(0.008)	(0.008)	(0.004)
(3) Average	0.4209	0.3939	0.3622	0.3625	0.3318	0.2980	0.4886	0.4497	0.4062	0.4089	0.3738	0.3317	0.3845	0.3508	0.3208	0.3173	0.2794	0.2544
IMCD	0.4240	0.3974	0.3679	0.3733	0.3532	0.3189	0.4971	0.4550	0.4144	0.4226	0.3976	0.3556	0.3858	0.3525	0.3247	0.3252	0.2973	0.2717
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
(4) Average	0.4198	0.3920	0.3616	0.3592	0.3264	0.2952	0.4856	0.4473	0.4048	0.4044	0.3668	0.3274	# 0.3847	0.3499	0.3214	0.3164	0.2781	0.2549

Lediot and Wolf denotes the single shrinkage targeting matrix method in Lediot and Wolf (2003, 2004). I, M,C and D denote the shrinkage towards indentity, market, constant correlation and diagonal matrices, respectively. Multivaraite targeting matrices consist of any combinations o f these shrinkage targeting matrices. [a] Constrainted portfolio with the annully targeting expected return, 300 basis points; [b] Constrainted portfolio with the annully targeting expected return, 800 basis points; [c] Constrainted portfolio with the annully targeting expected return, 1200 basis points. [d] Constrainted portfolio with the annully targeting expected return, 1600 basis points. [e] Constrainted portfolio with the annully targeting expected return, 2000 basis points. [f] GMV represents a global minimum variance portfolio. n represents the portfolio size. p values are reported in parenthesis. (1) average the standard deviations of the single targeting methods; (2) average the two targeting matrix methods; (3) average the three targeting matrix methods; (4) average all multivariate targeting methods. Standard devaiton is annulized by multiplying 12.

Table 2: Information Ratio Robust Tests of Out-of-Sample Portfolio Forecasting

	[a]: 300bp						[b]: 800bp						[c]: 1200bp					
	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500
Lediot and Wolf																		
I	0.2650	0.2879	0.2998	0.2919	0.2387	0.2594	0.2851	0.3012	0.2695	0.2603	0.2068	0.2009	0.2765	0.2857	0.2262	0.2155	0.1687	0.1469
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
M	0.2531	0.2673	0.2781	0.2657	0.2717	0.3255	0.2711	0.2782	0.2605	0.2496	0.2512	0.2844	0.2613	0.2618	0.2280	0.2182	0.2182	0.2363
	(0.011)	(0.007)	(0.004)	(0.002)	(0.006)	(0.009)	(0.007)	(0.005)	(0.160)	(0.009)	(0.000)	(0.001)	(0.003)	(0.008)	(0.057)	(0.026)	(0.003)	(0.007)
C	0.2489	0.2598	0.2650	0.2455	0.2240	0.2596	0.2640	0.2613	0.2324	0.2106	0.1743	0.1882	0.2493	0.2376	0.1909	0.1683	0.1276	0.1267
	(0.036)	(0.005)	(0.004)	(0.002)	(0.475)	(0.102)	(0.003)	(0.006)	(0.009)	(0.000)	(0.045)	(0.331)	(0.035)	(0.004)	(0.009)	(0.041)	(0.037)	(0.070)
D	0.2425	0.2704	0.2812	0.2741	0.2597	0.3169	0.2650	0.2854	0.2580	0.2480	0.2410	0.2683	0.2619	0.2713	0.2211	0.2079	0.2095	0.2150
	(0.005)	(0.006)	(0.006)	(0.002)	(0.003)	(0.000)	(0.003)	(0.014)	(0.092)	(0.137)	(0.008)	(0.005)	(0.044)	(0.004)	(0.265)	(0.218)	(0.006)	(0.004)
(1) Average	0.2523	0.2714	0.2810	0.2693	0.2485	0.2904	0.2713	0.2815	0.2551	0.2421	0.2183	0.2355	0.2622	0.2641	0.2166	0.2025	0.1810	0.1812
Generalized Multivariate Shrinkage																		
IM	0.2704	0.2882	0.2975	0.2856	0.2769	0.3197	0.2890	0.3006	0.2786	0.2675	0.2548	0.2746	0.2772	0.2845	0.2440	0.2331	0.2206	0.2244
	(0.005)	(0.072)	(0.619)	(0.077)	(0.002)	(0.003)	(0.004)	(0.506)	(0.004)	(0.049)	(0.005)	(0.008)	(0.027)	(0.111)	(0.000)	(0.003)	(0.006)	(0.005)
IC	0.2710	0.2864	0.2931	0.2742	0.2353	0.2439	0.2890	0.2937	0.2632	0.2420	0.1873	0.1738	0.2727	0.2724	0.2211	0.1989	0.1411	0.1153
	(0.007)	(0.006)	(0.903)	(0.051)	(0.060)	(0.012)	(0.001)	(0.785)	(0.229)	(0.010)	(0.049)	(0.064)	(0.023)	(0.002)	(0.065)	(0.080)	(0.001)	(0.007)
ID	0.2618	0.2856	0.3006	0.2894	0.2499	0.2736	0.2830	0.3020	0.2750	0.2621	0.2238	0.2177	0.2755	0.2886	0.2349	0.2201	0.1889	0.1645
	(0.001)	(0.056)	(0.741)	(0.001)	(0.007)	(0.002)	(0.513)	(0.098)	(0.002)	(0.025)	(0.008)	(0.006)	(0.263)	(0.030)	(0.009)	(0.019)	(0.007)	(0.008)
MC	0.2514	0.2546	0.2607	0.2415	0.2292	0.2610	0.2659	0.2569	0.2301	0.2112	0.1809	0.1913	0.2499	0.2341	0.1898	0.1716	0.1342	0.1306
	(0.073)	(0.002)	(0.007)	(0.002)	(0.764)	(0.059)	(0.000)	(0.004)	(0.043)	(0.001)	(0.030)	(0.246)	(0.020)	(0.004)	(0.001)	(0.032)	(0.008)	(0.041)
MD	0.2564	0.2718	0.2821	0.2702	0.2722	0.3272	0.2752	0.2825	0.2639	0.2535	0.2521	0.2851	0.2656	0.2657	0.2307	0.2212	0.2191	0.2361
	(0.096)	(0.004)	(0.053)	(0.002)	(0.004)	(0.004)	(0.000)	(0.086)	(0.398)	(0.414)	(0.003)	(0.002)	(0.004)	(0.006)	(0.004)	(0.006)	(0.007)	(0.008)
CD	0.2526	0.2643	0.2701	0.2521	0.2339	0.2651	0.2688	0.2688	0.2403	0.2210	0.1854	0.1942	0.2545	0.2468	0.2001	0.1801	0.1383	0.1324
	(0.003)	(0.007)	(0.008)	(0.007)	(0.759)	(0.021)	(0.059)	(0.015)	(0.004)	(0.006)	(0.829)	(0.433)	(0.035)	(0.025)	(0.010)	(0.002)	(0.002)	(0.964)
(2) Average	0.2606	0.2752	0.2840	0.2688	0.2496	0.2817	0.2785	0.2841	0.2585	0.2429	0.2141	0.2228	0.2659	0.2653	0.2201	0.2042	0.1737	0.1672
IMC	0.2720	0.2847	0.2874	0.2649	0.2362	0.2522	0.2888	0.2907	0.2589	0.2348	0.1883	0.1824	0.2713	0.2681	0.2174	0.1934	0.1415	0.1227
	(0.035)	(0.051)	(0.818)	(0.003)	(0.074)	(0.557)	(0.001)	(0.840)	(0.192)	(0.023)	(0.080)	(0.058)	(0.925)	(0.002)	(0.124)	(0.033)	(0.018)	(0.007)
IMD	0.2791	0.2971	0.3065	0.2886	0.2776	0.3250	0.2973	0.3083	0.2898	0.2738	0.2562	0.2812	0.2833	0.2908	0.2552	0.2409	0.2221	0.2311
	(0.001)	(0.000)	(0.020)	(0.078)	(0.009)	(0.002)	(0.006)	(0.012)	(0.022)	(0.029)	(0.002)	(0.000)	(0.007)	(0.011)	(0.002)	(0.000)	(0.001)	(0.000)
ICD	0.2787	0.2884	0.2949	0.2759	0.2314	0.2393	0.2957	0.2946	0.2675	0.2459	0.1832	0.1682	0.2771	0.2721	0.2258	0.2033	0.1367	0.1090
	(0.001)	(0.665)	(0.885)	(0.020)	(0.719)	(0.082)	(0.007)	(0.795)	(0.083)	(0.092)	(0.046)	(0.014)	(0.079)	(0.057)	(0.150)	(0.018)	(0.008)	(0.009)
MCD	0.2571	0.2636	0.2671	0.2460	0.2317	0.2654	0.2731	0.2667	0.2381	0.2168	0.1841	0.1960	0.2573	0.2434	0.1978	0.1773	0.1374	0.1349
	(0.340)	(0.007)	(0.010)	(0.003)	(0.318)	(0.029)	(0.207)	(0.005)	(0.022)	(0.004)	(0.064)	(0.108)	(0.079)	(0.000)	(0.044)	(0.004)	(0.008)	(0.810)
(3) Average	0.2717	0.2834	0.2890	0.2688	0.2442	0.2705	0.2887	0.2901	0.2636	0.2428	0.2029	0.2069	0.2723	0.2686	0.2240	0.2037	0.1594	0.1494
IMCD	0.2838	0.2914	0.2877	0.2633	0.2307	0.2422	0.3002	0.2960	0.2613	0.2365	0.1864	0.1772	0.2794	0.2714	0.2199	0.1965	0.1416	0.1203
	(0.009)	(0.006)	(0.098)	(0.000)	(0.089)	(0.004)	(0.006)	(0.518)	(0.221)	(0.066)	(0.064)	(0.022)	(0.037)	(0.049)	(0.241)	(0.096)	(0.001)	(0.034)
(4) Average	0.2668	0.2796	0.2861	0.2683	0.2459	0.2740	0.2842	0.2874	0.2606	0.2423	0.2075	0.2129	0.2694	0.2671	0.2215	0.2033	0.1656	0.1565

continued to Table2	[d]: 1600bp						[e]: 2000bp						[f]: GMV					
	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500	N=30	N=50	N=80	N=100	N=225	N=500
Lediot and Wolf																		
I	0.2356	0.2465	0.1745	0.1623	0.1259	0.0930	0.1793	0.1984	0.1243	0.1112	0.0852	0.0437	0.2798	0.2851	0.2413	0.2257	0.1874	0.1655
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
M	0.2221	0.2248	0.1850	0.1756	0.1769	0.1824	0.1702	0.1813	0.1407	0.1315	0.1355	0.1302	0.2661	0.2645	0.2469	0.2339	0.2464	0.2654
	(0.017)	(0.009)	(0.003)	(0.027)	(0.000)	(0.003)	(0.060)	(0.009)	(0.008)	(0.002)	(0.008)	(0.007)	(0.009)	(0.009)	(0.018)	(0.039)	(0.003)	(0.006)
C	0.2080	0.1988	0.1458	0.1239	0.0845	0.0710	0.1586	0.1584	0.1047	0.0847	0.0488	0.0245	0.2486	0.2452	0.2267	0.2018	0.1932	0.2003
	(0.005)	(0.001)	(0.009)	(0.007)	(0.006)	(0.098)	(0.004)	(0.002)	(0.037)	(0.042)	(0.001)	(0.028)	(0.009)	(0.004)	(0.081)	(0.006)	(0.030)	(0.041)
D	0.2298	0.2346	0.1752	0.1585	0.1699	0.1572	0.1818	0.1904	0.1294	0.1099	0.1304	0.1026	0.2582	0.2653	0.2401	0.2194	0.2357	0.2470
	(0.845)	(0.048)	(0.092)	(0.034)	(0.008)	(0.005)	(0.066)	(0.004)	(0.008)	(0.052)	(0.008)	(0.007)	(0.007)	(0.008)	(0.067)	(0.084)	(0.004)	(0.000)
(1) Average	0.2239	0.2262	0.1701	0.1551	0.1393	0.1259	0.1725	0.1821	0.1248	0.1093	0.1000	0.0753	0.2632	0.2650	0.2388	0.2202	0.2157	0.2195
Generalized Multivariate Shrinkage																		
IM	0.2328	0.2458	0.1979	0.1866	0.1781	0.1694	0.1750	0.1990	0.1500	0.1386	0.1355	0.1168	0.2860	0.2889	0.2620	0.2488	0.2452	0.2498
	(0.251)	(0.612)	(0.007)	(0.000)	(0.007)	(0.001)	(0.091)	(0.079)	(0.009)	(0.001)	(0.008)	(0.000)	(0.009)	(0.068)	(0.002)	(0.007)	(0.003)	(0.009)
IC	0.2246	0.2319	0.1727	0.1515	0.0973	0.0623	0.1674	0.1871	0.1272	0.1084	0.0602	0.0180	0.2742	0.2786	0.2502	0.2288	0.1960	0.1765
	(0.064)	(0.082)	(0.097)	(0.717)	(0.014)	(0.006)	(0.099)	(0.350)	(0.003)	(0.081)	(0.003)	(0.028)	(0.046)	(0.828)	(0.007)	(0.031)	(0.007)	(0.084)
ID	0.2358	0.2505	0.1849	0.1685	0.1480	0.1105	0.1806	0.2030	0.1353	0.1179	0.1081	0.0608	0.2767	0.2877	0.2503	0.2315	0.2083	0.1837
	(0.095)	(0.001)	(0.005)	(0.008)	(0.008)	(0.000)	(0.005)	(0.007)	(0.006)	(0.002)	(0.007)	(0.005)	(0.037)	(0.006)	(0.002)	(0.001)	(0.003)	(0.008)
MC	0.2062	0.1956	0.1452	0.1287	0.0902	0.0753	0.1544	0.1547	0.1043	0.0901	0.0533	0.0293	0.2498	0.2426	0.2213	0.2030	0.1944	0.2010
	(0.008)	(0.014)	(0.095)	(0.001)	(0.003)	(0.086)	(0.005)	(0.006)	(0.039)	(0.047)	(0.043)	(0.088)	(0.008)	(0.003)	(0.342)	(0.010)	(0.024)	(0.006)
MD	0.2259	0.2278	0.1867	0.1775	0.1776	0.1812	0.1730	0.1833	0.1415	0.1322	0.1360	0.1283	0.2699	0.2681	0.2493	0.2367	0.2472	0.2657
	(0.080)	(0.031)	(0.009)	(0.009)	(0.004)	(0.008)	(0.086)	(0.005)	(0.047)	(0.007)	(0.006)	(0.003)	(0.088)	(0.023)	(0.022)	(0.021)	(0.006)	(0.001)
CD	0.2123	0.2084	0.1552	0.1359	0.0942	0.0761	0.1612	0.1673	0.1137	0.0962	0.0573	0.0292	0.2518	0.2530	0.2331	0.2120	0.2019	0.2046
	(0.022)	(0.005)	(0.040)	(0.034)	(0.001)	(0.193)	(0.009)	(0.006)	(0.425)	(0.079)	(0.064)	(0.071)	(0.002)	(0.006)	(0.057)	(0.763)	(0.009)	(0.039)
(2) Average	0.2229	0.2267	0.1738	0.1581	0.1309	0.1125	0.1686	0.1824	0.1287	0.1139	0.0917	0.0637	0.2681	0.2698	0.2444	0.2268	0.2155	0.2136
IMC	0.2219	0.2263	0.1694	0.1474	0.0971	0.0686	0.1638	0.1805	0.1243	0.1054	0.0596	0.0236	0.2748	0.2770	0.2471	0.2235	0.1971	0.1866
	(0.009)	(0.020)	(0.086)	(0.060)	(0.007)	(0.034)	(0.000)	(0.001)	(0.064)	(0.259)	(0.010)	(0.007)	(0.076)	(0.288)	(0.001)	(0.174)	(0.049)	(0.002)
IMD	0.2356	0.2502	0.2069	0.1940	0.1789	0.1750	0.1749	0.2013	0.1558	0.1443	0.1350	0.1207	0.2924	0.2994	0.2725	0.2583	0.2453	0.2526
	(0.096)	(0.015)	(0.006)	(0.007)	(0.000)	(0.005)	(0.078)	(0.720)	(0.007)	(0.007)	(0.009)	(0.009)	(0.009)	(0.007)	(0.000)	(0.007)	(0.009)	(0.008)
ICD	0.2261	0.2302	0.1765	0.1553	0.0926	0.0557	0.1666	0.1843	0.1298	0.1113	0.0554	0.0112	0.2791	0.2831	0.2550	0.2351	0.1916	0.1698
	(0.09)	(0.06)	(0.07)	(0.05)	(0.00)	(0.00)	(0.08)	(0.05)	(0.00)	(0.06)	(0.00)	(0.00)	(0.07)	(0.09)	(0.08)	(0.00)	(0.04)	(0.09)
MCD	0.2125	0.2033	0.1525	0.1340	0.0931	0.0791	0.1592	0.1608	0.1105	0.0949	0.0560	0.0325	0.2555	0.2520	0.2292	0.2081	0.1976	0.2053
	(0.03)	(0.00)	(0.00)	(0.04)	(0.01)	(0.88)	(0.01)	(0.00)	(0.80)	(0.09)	(0.00)	(0.45)	(0.02)	(0.00)	(0.19)	(0.03)	(0.02)	(0.00)
(3) Average	0.2240	0.2275	0.1763	0.1577	0.1154	0.0946	0.1661	0.1817	0.1301	0.1140	0.0765	0.0470	0.2754	0.2779	0.2510	0.2312	0.2079	0.2036
IMCD	0.2247	0.2269	0.1704	0.1501	0.0977	0.0677	0.1623	0.1786	0.1234	0.1070	0.0598	0.0231	0.2830	0.2861	0.2477	0.2272	0.1919	0.1765
	(0.07)	(0.02)	(0.03)	(0.09)	(0.04)	(0.04)	(0.02)	(0.05)	(0.00)	(0.19)	(0.00)	(0.04)	(0.04)	(0.05)	(0.03)	(0.27)	(0.06)	(0.03)
(4) Average	0.2235	0.2270	0.1744	0.1572	0.1223	0.1019	0.1671	0.181813	0.1287	0.1133	0.0833	0.0540	0.2721	0.2742	0.2471	0.2285	0.2106	0.2066

Lediot and Wolf denotes the single shrinkage targeting matrix method in Lediot and Wolf (2003, 2004). I, M,C and D denote the shrinkage towards indentity, market, constant correlation and diagonal matrices, respectively. Multivaraite targeting matrices consist of any combinations o f these shrinkage targeting matrices. [a] Constrainted portfolio with the annully targeting expected return, 300 basis points; [b] Constrainted portfolio with the annully targeting expected return, 800 basis points; [c] Constrainted portfolio with the annully targeting expected return, 1200 basis points. [d] Constrainted portfolio with the annully targeting expected return, 1600 basis points. [e] Constrainted portfolio with the annully targeting expected return, 2000 basis points. [f] GMV represents a global minimum variance portfolio. n represents the portfolio size. p values are reported in parenthesis. (1) average the standard deviations of the single targeting methods; (2) average the two targeting matrix methods; (3) average the three targeting matrix methods; (4) average all multivariate targeting methods.

To measure the statistical significance of out-of-sample performance, I use bootstrapping methods. In particular, to compute the *p*-values for the information ratios I apply the bootstrapping method proposed by Ledoit and Wolf (2008), while to test the hypothesis of the equality of two given portfolios' variances, I employ the stationary bootstrap of Politis and Romano (1994), and then the resulting bootstrap p-values are generated by the methodology suggested in Ledoit and Wolf (2008, Remark 3.2). The programming code for the robust tests of Ledoit and Wolf (2008) is available at http://www.econ.uzh.ch/faculty/wolf/publications.html.

Table 1 reports the out-of-sample standard deviations for the different portfolios and p-values in parenthesis. The standard deviations of portfolios estimated from the single targeting of identity matrix are set as the benchmark, so that all other models are compared to it in terms of the equality tests. The difference is significant between two portfolios if the *p*-value is less than 5%. I denote any p-values less than 1% by 0.00. All other pair-wise p-values are also computed and used for discussion, while I do not report those due to the space limit. The averages of portfolio standard deviations are also reported.

Among the single targetings of the mean-variance and global minimum portfolios, shrinkages towards market model (M) have the lowest standard deviations, while shrinkages towards constant correlation (C) are the best for the portfolios with $q = 3\%$. In contrast, among the various multivariate shrinkages, shrinkages towards the combination of market model and diagonal matrix consistently achieve the lowest standard deviations, while shrinkages towards CD and MCD have better performance for the portfolios with $q = 3\%$. Hence, comparing the best multivariate shrinkage, MD, to the best single targeting shrinkage, M, it can be observed that MD always has obtained the larger reduction in portfolio standard deviations than those of M across different portfolio types and sizes. Importantly, the differences between MD and M are statistically significant.

On average, the multivariate shrinkage targeting have the lower standard deviations for the global minimum portfolios and the portfolios with *q* at modest levels, such as 3%-12%. However, the single targeting performs better when the portfolios were required by a relative higher level of conditional expected returns, i.e., $q = 16\%, 20\%$. This evidence is in line with the findings of Kirdy and Ostdiek (2012) that a high targeting conditional expected excess returns might lead to poor out-of-sample performance because it greatly magnifies both estimation risk and portfolio turnover.

Table 2 presents the empirical results for the out-of-sample information ratios and p-values of the corresponding equality tests. Among the single targetings, shrinkages towards identity matrix (I) have the highest information ratios for the small and medium portfolios, such as $N = 30, 80, 100$, while shrinkages towards market model (M) obtain the highest information ratios for the large sized portfolios. Among the multivariate shrinkage estimations, I find that IMD achieves the highest information ratios for the portfolio sizes from 30 to 225, while MD get the best for the portfolios of size, 500. Similarly, comparing the best multivariate shrinkage to the best single targeting, IMD and MD consistently outperform I and M across different portfolio types, sizes and q constraints. Also, the differences between multivariate shrinkage and single targeting are statistically significant as indicated by *p*-values which are generally smaller than 5%.

However, on average, single targetings have higher information ratios than those of multivariate shrinkages for the large sized portfolios, e.g., $N = 225, 500$. The multivariate shrinkages appear the advantages of information ratios for the small and medium sized portfolios of both mean-variance and global minimum-variance. This result might imply that for a large sized portfolio, despite that a multivariate shrinkage brings more information to estimating shrinkage intensities; it also brings noises into shrinkage intensity.

Table 3: Portfolio Turnovers

	[a]: 300bp						[b]: 800bp						[c]: 1200bp						[d]: 1600bp						[e]: 2000bp						[f]: GMV					
	n=30	n=50	n=80	n=100	n=225	n=500	n=30	n=50	n=80	n=100	n=225	n=500	n=30	n=50	n=80	n=100	n=225	n=500	n=30	n=50	n=80	n=100	n=225	n=500	n=30	n=50	n=80	n=100	n=225	n=500	n=30	n=50	n=80	n=100	n=225	n=500
Lediot and Wolf																																				
I	0.407	0.420	0.463	0.492	0.561	0.573	0.277	0.310	0.364	0.400	0.492	0.519	0.214	0.262	0.329	0.370	0.479	0.510	0.224	0.273	0.346	0.388	0.505	0.536	0.293	0.334	0.407	0.448	0.566	0.591	0.181	0.245	0.320	0.366	0.476	0.506
M	0.416	0.413	0.435	0.451	0.483	0.543	0.283	0.302	0.332	0.354	0.410	0.477	0.233	0.267	0.308	0.332	0.406	0.474	0.264	0.300	0.346	0.370	0.453	0.519	0.344	0.376	0.425	0.448	0.535	0.598	0.199	0.245	0.294	0.320	0.397	0.466
C	0.419	0.420	0.440	0.454	0.508	0.589	0.292	0.318	0.350	0.371	0.463	0.555	0.246	0.292	0.339	0.364	0.482	0.577	0.273	0.325	0.383	0.409	0.543	0.636	0.346	0.396	0.460	0.488	0.630	0.722	0.207	0.263	0.316	0.345	0.456	0.550
D	0.437	0.459	0.510	0.547	0.614	0.631	0.298	0.342	0.407	0.449	0.541	0.566	0.241	0.303	0.382	0.426	0.534	0.563	0.266	0.331	0.418	0.459	0.581	0.606	0.344	0.405	0.499	0.537	0.666	0.684	0.209	0.287	0.371	0.419	0.529	0.555
(1) Average	0.420	0.428	0.462	0.486	0.541	0.584	0.288	0.318	0.363	0.394	0.476	0.529	0.233	0.281	0.340	0.373	0.475	0.531	0.257	0.307	0.373	0.407	0.520	0.574	0.332	0.378	0.448	0.480	0.599	0.649	0.199	0.260	0.325	0.362	0.464	0.519
Generalized Multivariate Shrinkage																																				
IM	0.397	0.397	0.424	0.441	0.476	0.533	0.268	0.288	0.322	0.345	0.403	0.469	0.208	0.244	0.289	0.317	0.393	0.462	0.228	0.265	0.316	0.345	0.430	0.500	0.301	0.333	0.387	0.414	0.501	0.571	0.175	0.225	0.278	0.308	0.387	0.456
IC	0.399	0.396	0.416	0.429	0.467	0.543	0.274	0.292	0.322	0.342	0.416	0.507	0.215	0.252	0.296	0.321	0.422	0.521	0.230	0.270	0.324	0.352	0.467	0.568	0.296	0.331	0.391	0.419	0.541	0.642	0.174	0.224	0.277	0.305	0.405	0.501
ID	0.411	0.424	0.470	0.500	0.566	0.581	0.277	0.311	0.368	0.405	0.495	0.523	0.213	0.263	0.333	0.375	0.482	0.515	0.226	0.278	0.353	0.396	0.512	0.545	0.298	0.343	0.420	0.460	0.577	0.607	0.179	0.247	0.324	0.370	0.480	0.511
MC	0.413	0.408	0.425	0.433	0.456	0.525	0.286	0.305	0.331	0.345	0.402	0.486	0.240	0.278	0.316	0.336	0.419	0.508	0.270	0.311	0.359	0.381	0.480	0.570	0.343	0.383	0.437	0.460	0.568	0.658	0.200	0.245	0.291	0.312	0.391	0.481
MD	0.414	0.411	0.435	0.451	0.487	0.547	0.280	0.299	0.331	0.353	0.414	0.481	0.228	0.263	0.306	0.330	0.410	0.478	0.257	0.295	0.342	0.367	0.456	0.523	0.336	0.370	0.422	0.445	0.538	0.601	0.194	0.241	0.291	0.318	0.401	0.470
CD	0.414	0.413	0.432	0.445	0.488	0.568	0.286	0.308	0.338	0.357	0.438	0.532	0.237	0.278	0.323	0.346	0.455	0.553	0.262	0.307	0.364	0.389	0.514	0.611	0.334	0.377	0.439	0.466	0.600	0.697	0.196	0.247	0.299	0.327	0.430	0.527
(2) Average	0.408	0.408	0.433	0.450	0.490	0.550	0.279	0.301	0.335	0.358	0.428	0.500	0.224	0.263	0.310	0.337	0.430	0.506	0.245	0.288	0.343	0.371	0.476	0.553	0.318	0.356	0.416	0.444	0.554	0.629	0.186	0.238	0.293	0.323	0.415	0.491
IMC	0.398	0.394	0.415	0.427	0.457	0.529	0.274	0.291	0.321	0.339	0.402	0.490	0.216	0.253	0.296	0.319	0.411	0.508	0.234	0.274	0.327	0.354	0.462	0.563	0.301	0.337	0.395	0.424	0.542	0.644	0.177	0.224	0.275	0.301	0.389	0.484
IMD	0.389	0.382	0.403	0.413	0.439	0.489	0.260	0.273	0.301	0.317	0.364	0.421	0.199	0.227	0.264	0.285	0.348	0.409	0.217	0.246	0.287	0.310	0.380	0.442	0.290	0.313	0.357	0.379	0.448	0.509	0.164	0.205	0.251	0.275	0.341	0.404
ICD	0.390	0.384	0.403	0.414	0.450	0.521	0.267	0.282	0.310	0.327	0.395	0.483	0.207	0.240	0.282	0.304	0.399	0.495	0.220	0.258	0.309	0.334	0.442	0.541	0.286	0.318	0.375	0.400	0.514	0.613	0.165	0.209	0.261	0.287	0.382	0.477
MCD	0.411	0.405	0.422	0.430	0.456	0.530	0.285	0.301	0.327	0.342	0.402	0.492	0.238	0.272	0.311	0.330	0.419	0.513	0.265	0.304	0.353	0.374	0.480	0.574	0.337	0.375	0.430	0.452	0.567	0.662	0.197	0.239	0.285	0.307	0.392	0.486
(3) Average	0.397	0.391	0.411	0.421	0.450	0.517	0.271	0.287	0.315	0.331	0.391	0.471	0.215	0.248	0.288	0.310	0.394	0.481	0.234	0.270	0.319	0.343	0.441	0.530	0.303	0.336	0.389	0.414	0.518	0.607	0.176	0.219	0.268	0.292	0.376	0.463
IMCD	0.387	0.378	0.396	0.402	0.416	0.469	0.265	0.275	0.301	0.313	0.354	0.424	0.206	0.235	0.272	0.290	0.356	0.436	0.222	0.255	0.300	0.321	0.402	0.486	0.288	0.316	0.366	0.388	0.476	0.561	0.165	0.203	0.248	0.268	0.335	0.416
(4) Average	0.402	0.399	0.422	0.435	0.469	0.530	0.275	0.293	0.325	0.344	0.408	0.483	0.219	0.255	0.299	0.323	0.410	0.491	0.239	0.278	0.330	0.357	0.457	0.538	0.310	0.345	0.402	0.428	0.534	0.615	0.180	0.228	0.280	0.307	0.394	0.474

Lediot and Wolf denotes the single shrinkage targeting matrix method in Lediot and Wolf (2003, 2004). I, M, C and D denote the shrinkage targeting to indentity, market, constant correlation and diagonal matrice, respectively. Multivaraite targeting matrice consist of any combinationo f these shrinkage targeting matrix. [a] Constrainted portfolio with the annully targeting expected return, 300 basis points; [b] Constrainted portfolio with the annully targeting expected return, 800 basis points; [c] Constrainted portfolio with the annully targeting expected return, 1200 basis points. [d] Constrainted portfolio with the annully targeting expected return, 1600 basis points. [e] Constrainted portfolio with the annully targeting expected return, 2000 basis points. [f] GMV represents a global minimum variance portfolio. n represents the portfolio size. (1) average the turnovers of the single targeting methods; (2) average the two targeting matrix methods; (3) average the three targeting matrix methods; (4) average all multivariate targeting methods.

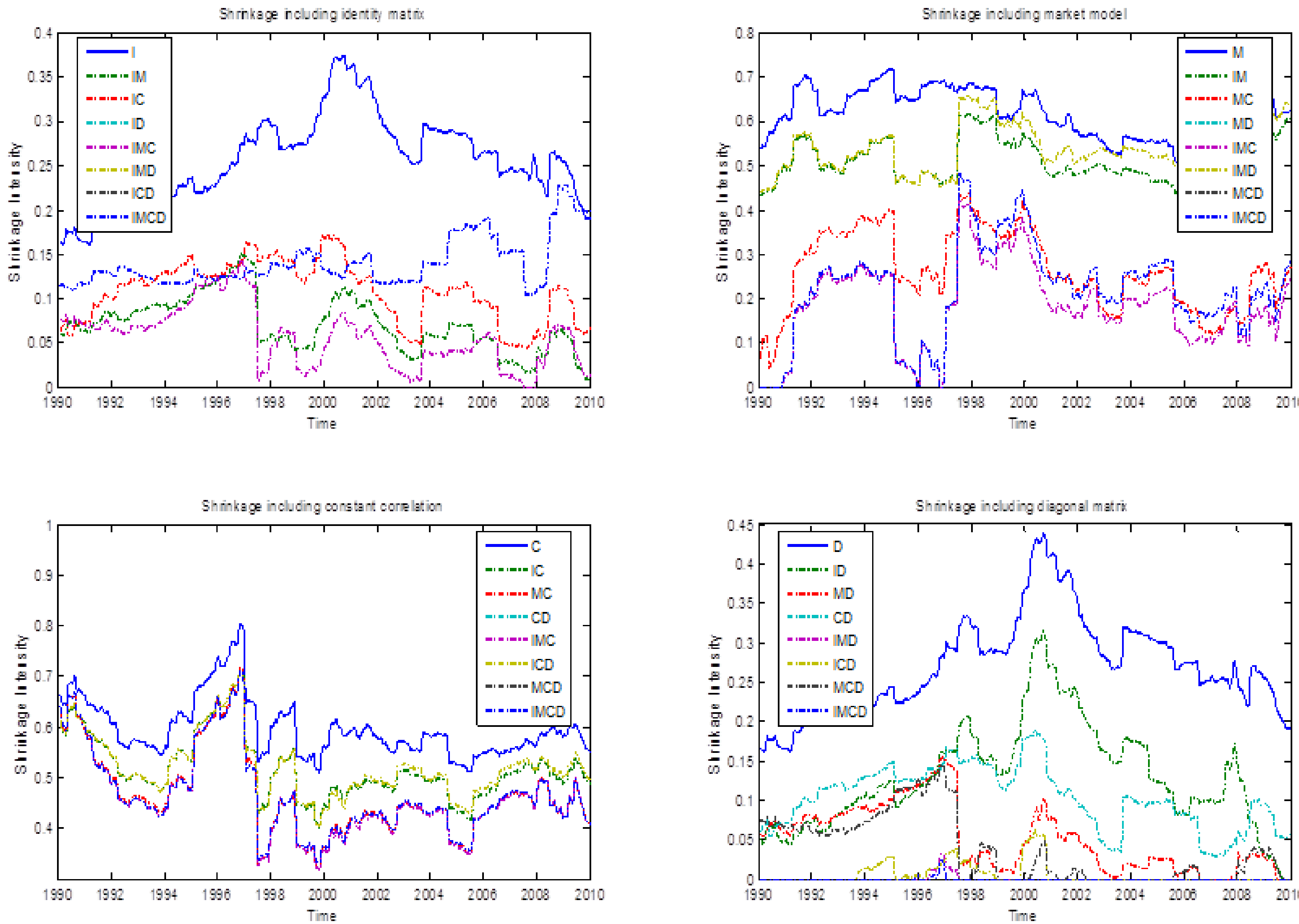


Figure 1. Shrinkage intensity

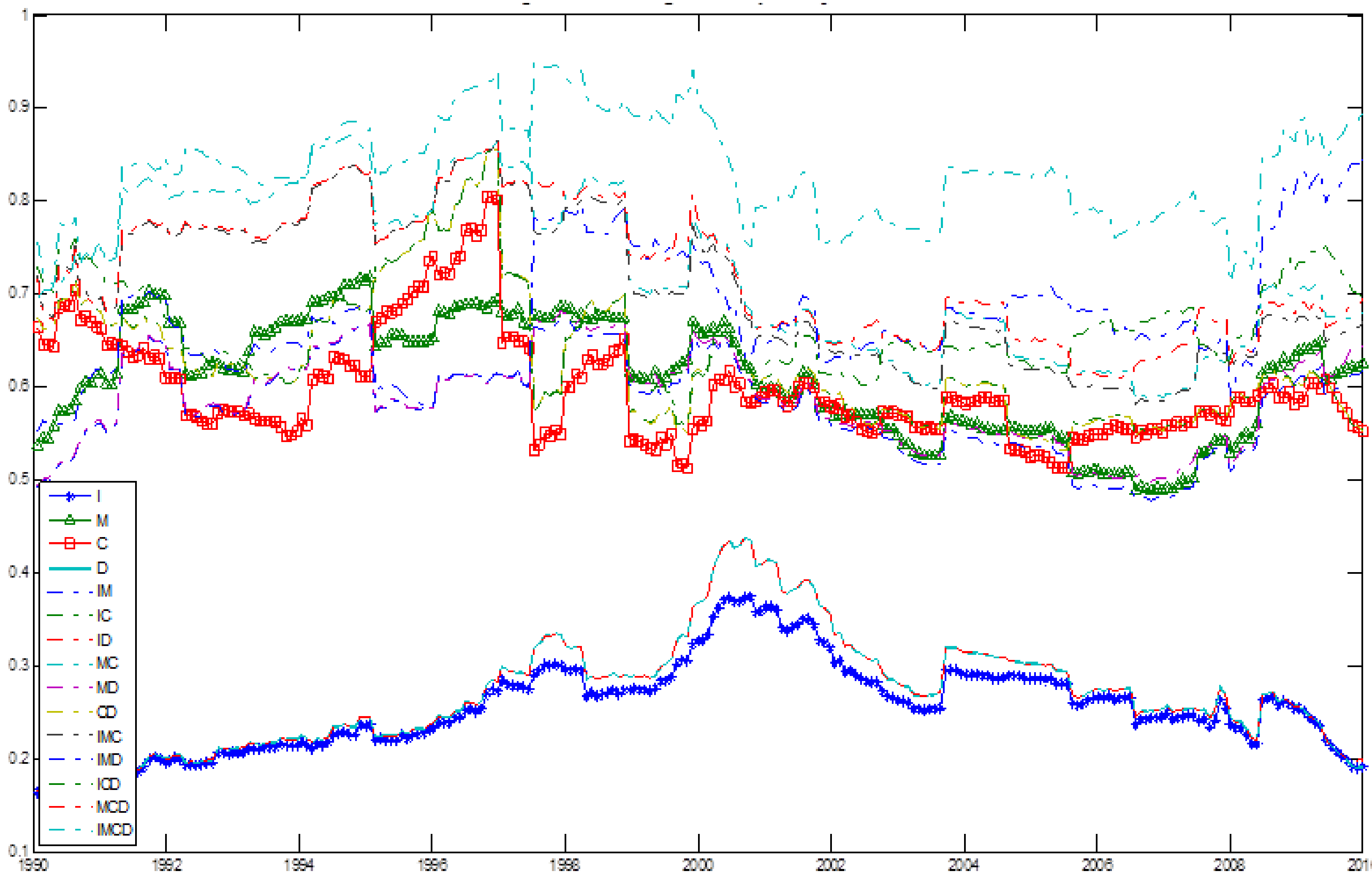


Figure 2. Shrinkage intensity comparison

Portfolio turnovers in Table 3 show that shrinkages towards identity (I) and market model (M) have the lowest turnovers among the single targetings. By contrast, the multivariate shrinkage of IMD has the lowest turnovers across the portfolios, while $IMCD$ obtains the lowest for the global minimum-variance portfolio and the mean-variance portfolio with $q = 3\%$, and some of $q = 8\%$. Comparing the best multivariate shrinkages to the best single targetings, IMD and $IMCD$ always have lower turnovers than I and M. Observably on averag, all multivariate shrinkages have lower turnovers than single targetings.

Additionally, there are many cases among the estimations which show that a multivariate shrinkage performs better than any single component shrinkage of that multivariate shrinkage. For instance, in terms of the standard deviation, the multivariate shrinkage of CD for the portfolio size $N = 30$ and $q = 8\%$ has the standard deviation, 0.4112, smaller than those of both the single targetings of C, 0.415, and D, 0.4326; moreover, in terms of the information ratio, the multivariate shrinkage of ICD for the portfolio size $N = 30$ and $q = 8\%$ has the information ratio, 0.2957, larger than those of the single component targetings of I, 0.2851, C, 0.264, and D, 0.265; looking at portfolio turnovers, we see that the multivariate shrinkage of $IMCD$ for the portfolio size $N = 30$ and $q = 8\%$ has the turnover, 0.265, less than that of any of a single component, $I, M, C, D,$ 0.277, 0.283, 0.292, and 0.298, respectively.

Additionally, it is also interesting to see the difference between the shrinkage intensities of the multivariate shrinkage and the single targetings. Figure 1 plots the shrinkage intensities of the portfolio with the size $n = 225$ and $q = 12\%$. All other portfolios have the similar graphical results. We take one of the single targeting matrices compared to those multivariate shrinkages which contain that single targeting matrix as a component. It can be seen that the estimated shrinkage intensities vary significantly for different targeting matrices. For instance, shrinkage intensity of the single targeting market model is around 0.65, while the multivariate shrinkages including market model has the lowest around 0.2 on average. Moreover, the comparison shows that the shrinkage intensities estimated by multivariate shrinkages always are lower than those of single targetings.

Figure 2 plots the shrinkage intensities of the sum of the components' shrinkage intensities of a multivariate shrinkage, compared to those of single targetings. For instance, the shrinkage intensity of MCD is computed by summing up the shrinkage intensities of M, C, and D which are estimated by the multivariate shrinkage of MCD. We see that the single targeting of I and D have the lowest shrinkage intensity in general, whereas most of multivariate shrinkages have the higher shrinkage intensity. In addition, the shrinkage intensity variance of multivariate shrinkage method is larger than single targetings, except for the single targeting of constant correlation.

9. Conclusion

This paper generalizes the single targeting shrinkage method of Ledoit and Wolf to a multivariate shrinkage setting. The optimal shrinkage intensity solution of the generalized multivariate shrinkage has also been provided in this paper. The mean-variance and global minimum-variance portfolios are constructed with various sizes, and constraint expected returns for out-of-sample portfolio performance.

Empirically, the generalized multivariate shrinkage outperforms over the single targeting method, in terms of reducing out-of-sample portfolio variance across different portfolio types, sizes, and conditional expected returns. It is also observed that the proposed multivariate shrinkage method has higher information ratios only for small and medium sized portfolios. Additionally, the out-of-sample portfolios of the generalized multivariate shrinkage appear more attractive to an active portfolio manager due to their lower portfolio turnovers than those of single targetings. In addition, there are many cases among the estimations which show that a multivariate shrinkage performs better than a single component shrinkage of that multivariate shrinkage.

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Appendix A

Proof for Optimal Generalized Multivariate Shrinkage Intensity

The similar objective function is used as in Ledoit and Wolf (2003, 2004a&b):

$$\min_{\alpha} E\left\|\Sigma^*(\alpha)-\Sigma\right\|^2$$

$$s.t \quad \Sigma^* = (\alpha \otimes I)' F + (1-\alpha'\mathbf{1})S$$

where $\|Z\|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} z_{ij}^2$ for a $N \times N$ matrix with the entries, $z_{ij}\,|_{i,j=1,\dots,N}$. F be a vector with K elements. Each of the elements in F is a $N \times N$ structure targeting matrix, such that $F = \left(f^1,\dots,f^K\right)'$. α is a dimension K weighting vector, $(\alpha_1,\dots,\alpha_K)'$ with $\alpha_k \in [0,1]$, $\sum_{k=1}^{K}\alpha_k \in [0,1]$. Also, $(\alpha \otimes I)' F = \sum_{k=1}^{K}\alpha_k f^k$, where I is an identity matrix with the dimension $N \times N$ and $\mathbf{1}$ denotes a comfortable vector of ones. $\Sigma^*(\alpha)$ is the linear combination of the multivariate shrinkage targeting matrices, F, and the sample covariance matrix, S. Σ is a $N \times N$ population covariance matrix.

The risk function of the objective function is derived as:

$$R(\alpha) = E[L(\alpha)]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\alpha' f_{ij} + (1-\alpha'\mathbf{1})s_{ij} - \sigma_{ij}\right]^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\alpha'(f_{ij}-\mathbf{1}s_{ij}) + s_{ij} - \sigma_{ij}\right]^2 \sum_{i=1}^{n}\sum_{j=1}^{n} var\left[\alpha'(f_{ij}-\mathbf{1}s_{ij}) + s_{ij}\right] + \left[E\left(\alpha' f_{ij} + (1-\mathbf{1}'\alpha)s_{ij} - \sigma_{ij}\right)\right]^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha' var(f_{ij}-\mathbf{1}s_{ij})\alpha + var(s_{ij}) + 2\alpha' cov(f_{ij}-\mathbf{1}s_{ij}, s_{ij}) + \left[E\left(\alpha'(f_{ij}-\mathbf{1}\sigma_{ij}) + (1-\alpha'\mathbf{1})(s_{ij}-\sigma_{ij})\right)\right]^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha' var(f_{ij} - \mathbf{1}s_{ij})\alpha + var(s_{ij}) + 2\alpha' cov(f_{ij} - \mathbf{1}s_{ij}, s_{ij}) + \left[\alpha'\left(E(f_{ij}) - \mathbf{1}\sigma_{ij}\right)\right]^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha' var(f_{ij} - \mathbf{1}s_{ij})\alpha + var(s_{ij}) + 2\alpha' cov(f_{ij} - \mathbf{1}s_{ij}, s_{ij}) + \left[\alpha'\left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)\right]^2$$

where f_{ij} is a dimension K vector with the elements $(f_{ij}^1, \ldots, f_{ij}^K)$. f_{ij}^k is the element of the i^{th} row and j^{th} column of the $n \times n$ matrix f^k, $k = 1, \ldots, K$. $E(s_{ij}) = \sigma_{ij}$ is a scalar , and $E(f_{ij}) = \phi_{ij}$ is a K dimension vector. σ_{ij} is the element of Σ at the i^{th} row and j^{th} column.

Minimize the risk function, $R(\alpha)$, with respect to α to get:

$$F.O.C.: \quad \frac{\partial R(\alpha)}{\partial \alpha} = 2\sum_{i=1}^{n}\sum_{j=1}^{n} var(f_{ij} - \mathbf{1}s_{ij})\alpha + cov(f_{ij} - \mathbf{1}s_{ij}, s_{ij}) + \left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)\left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)'\alpha$$

$$S.O.C.: \quad \frac{\partial^2 R(\alpha)}{\partial \alpha^2} = 2\sum_{i=1}^{n}\sum_{j=1}^{n} var\left(f_{ij} - \mathbf{1}s_{ij}\right) + \left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)\left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)'$$

Note that the second order condition is a positive definite such that there exists a solution for minimizing the equations (1) and (3).

Set the first order condition to be zero:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[var(f_{ij} - \mathbf{1}s_{ij}) + \left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)\left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)'\right]\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n} cov(\mathbf{1}s_{ij} - f_{ij}, s_{ij})$$

where

$$\left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)\left(\phi_{ij} - \mathbf{1}\sigma_{ij}\right)' = \begin{pmatrix} (\phi_{ij}^1 - \sigma_{ij})^2 & \cdots & (\phi_{ij}^1 - \sigma_{ij})(\phi_{ij}^K - \sigma_{ij}) \\ \vdots & \ddots & \vdots \\ (\phi_{ij}^K - \sigma_{ij})(\phi_{ij}^1 - \sigma_{ij}) & \cdots & (\phi_{ij}^K - \sigma_{ij})^2 \end{pmatrix}$$

is a $K \times K$ matrix, and

$$cov(f_{ij} - \mathbf{1}s_{ij}, s_{ij}) = \begin{pmatrix} cov(f_{ij}^1, s_{ij}) - var(s_{ij}) \\ \vdots \\ cov(f_{ij}^K, s_{ij}) - var(s_{ij}) \end{pmatrix}$$

is a $K \times 1$ vector. Solve for optimal α^*, the formula for the optimal multivariate shrinkage intensity can be obtained as in equation (6).

Appendix B.

Definitions

I take the case $F = (\mu I, C, M)$. C is a constant correlation targeting matrix, and M is a market model as in Ledoit and Wolf (2003). the following scalars are defined as: $\alpha_1^2 = \|\Sigma - \mu I\|^2$,

$\alpha_2^2 = \|\Sigma - M\|^2$, $\alpha_3^2 = \|\Sigma - C\|^2$, $\beta^2 = E\|S - \Sigma\|^2$, $\eta_1^2 = E\|S - \mu I\|^2$, $\eta_2^2 = E\|S - M\|^2$, $\eta_3^2 = E\|S - C\|^2$, $\mu =< \Sigma, I >$.

Lemma 1: (1) $\eta_1^2 = \alpha_1^2 + \beta^2$; (2) $\eta_2^2 = \alpha_2^2 + \beta^2$; (3) $\eta_3^2 = \alpha_3^2 + \beta^2$

Proof:

$$\eta_1^2 = E\|S - \mu I\|^2 = E\|S - \Sigma + \Sigma - \mu I\|^2$$

$$= \|\Sigma - \mu I\|^2 + E\|S - \Sigma\|^2 + 2E\left[< S - \Sigma, \Sigma - \mu I >\right]$$

$$= \|\Sigma - \mu I\|^2 + E\|S - \Sigma\|^2 + 2 < E[S - \Sigma], \Sigma - \mu I > \tag{11}$$

Notice that $E(S) = \Sigma$; therefore, the third term on the right-hand side of equation (11) is equal to zero. This completes the proof of Lemma 1(1). The proof is similar for (2) and (3).

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