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Modelling Alpha in a CAPM with Heterogenous Beliefs*

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Abstract

The alpha is one of the most used terms in finance. Yet, the alpha is mystical since it has no theory. It is, for example, in contradiction to the standard CAPM with homogenous beliefs. The purpose of this paper is to show that the alpha naturally arises in a financial market equilibrium when the CAPM is extended to heterogenous beliefs. We show that the hunt for alpha-opportunities is a zero-sum game and that alpha-opportunities erode with the assets under management. Moreover, it is shown that a positive alpha is not necessarily a good criterion for the choice between active and passive investment. Finally, we argue that the standard CAPM with homogenous beliefs can be seen as the long run outcome of our model when investors' expectations are linked to the trading success.

JEL Classifications : G11, G12, G14

Keywords: CAPM, heterogenous beliefs, active and passive investment

1. Introduction

The Capital Asset Pricing Model, CAPM, is a rich source of intuition and also the basis for many practical financial decisions. The asset pricing implication of the CAPM is the security market line, SML, according to which the excess return of any asset over the risk free rate is proportional to the excess return

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of the market portfolio over the risk free rate. The proportionality factor is the beta, i.e. the covariance of the asset's return to the return of the market portfolio divided by the variance of the market portfolio. The beta is the only risk factor that is rewarded according the CAPM. Hence, investors requiring a high expected return will have to accept a high beta. Some investors, however, want to achieve more. They claim to be able to achieve positive deviations of expected returns over those given by the SML. Those deviations of returns are referred to as Jensen's alpha or short as the "*alpha*". Indeed the alpha is nowadays a common term in the finance jargon. Hedge Funds for example consider themselves to be alpha generating strategies; many of them use the term "alpha" in their marketing brochures and some of them even as part of their name.[1] While many opinion leaders in the world of finance claim that the existence of alpha contradicts the validity of the CAPM, we argue in this paper that a simple extension of the CAPM towards heterogenous beliefs is already able to explain the alpha in a financial market equilibrium. The extension we use goes back to the CAPM with heterogenous beliefs suggested by Lintner (1969).

In this paper we first show how to derive the SML in a CAPM with heterogenous beliefs. With heterogenous beliefs investors hold heterogenous and under-diversified portfolios. Thus, the unrealistic[2] two-fund-separation property of the CAPM with homogenous beliefs according to which all investors hold the same portfolio of risky assets does not hold. Yet, on the level of the market the SML holds so that in regressions on equity returns as initiated by Fama and French (1992) no additional factors are needed. The derivation of the SML is an aggregation result since for the SML only the average belief and risk aversion matters. We show that in this aggregation individual beliefs get weighted by the risk aversion and the relative wealth of the investors. Thus the wealthier and the less risk averse an investor is the more his beliefs will determine the market beliefs. Then we define the alpha as the return an investor expects to get in excess of the risk adjusted market return. Based on the aggregation result underlying the SML with heterogenous beliefs we can then show that the hunt for alpha-opportunities is a zero-sum game and that alpha-opportunities erode with the assets under management. Moreover, we show that a positive alpha is not necessarily a good criterion for the choice between active and passive investment. Finally, based on a market selection argument we argue that the standard CAPM with homogenous beliefs can be seen as the long run outcome of our model when investors' expectations are linked to the trading success. To do so we extend our model by endogenizing agents' information by allowing them to be either passive, in which case they invest according to the average expectation embodied in the market returns, or to be active, in which case they can acquire superior information at some cost. In our model we show that the decision of being active or passive depends on the efficiency of the market, the quality of the investor's belief, his degree of risk aversion and of course the costs for being active. An investor is more inclined to be active the less efficient the market is, the better his information and the less risk averse he is. By contrast, it can be shown that expecting a positive alpha is not necessarily a good criterion for becoming active. We give simple examples pointing out that expecting a positive alpha from the active strategy is neither a necessary nor a sufficient condition for becoming active. In our model, delegating active investment to portfolio managers only makes sense if the performance fee increases with the skill of the portfolio manager and is bounded above by some function of the degree of inefficiency of the market. Our model provides new measures for both of these components. Then we derive the main criterion for active portfolio management based on the measures of market efficiency and the skill of the active managers. Furthermore we show which structure fees for active management should have.

The rest of the paper is organized as follows. The next section relates our results to the literature. Then in section 3 a formal description of the CAPM with heterogenous beliefs is given and the aggregation result of the SML is derived. Section 4 defines the alpha and derives the zero-sum property in the CAPM

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[1] To list some examples: Goldman Sachs offers "Global Alpha", Merill Lynch "Absolute Alpha Fund" and UBS "Alpha Hedge" and "Alpha Select".

[2] Canner, Mankiw and Weil (1997) were the first to point out that even well trained financial advisers do not recommend to follow the two-fund-separation property.

with heterogenous beliefs. In section 5 we derive the main criterion for active portfolio management and show that the CAPM with homogenous beliefs can be seen as the long-run outcome of our model.

2. Literature Review

Our first result, the derivation of the SML with heterogenous beliefs, parallels Chiarella, Dieci, and He (2006) and derives the security market line of the CAPM as an aggregation result without using the unrealistic two-fund-separation property. Chiarella *et al*. (2006) use different utility functions and also different endowments of the investors. We further explain the impact of these differences below. The security market line turns out to hold with respect to *average* beliefs about the expected asset returns and covariances of returns. However, under heterogenous expectations this security market line does not coincide with the individual security market lines defined with respect to investors' subjective beliefs. Hence, unlike in the CAPM with homogenous beliefs, investors in equilibrium will hold different portfolios of risky assets. In particular, the often observed feature of underdiversification (see, for example, Odean, 1999; Goetzmann & Kumar, 2008; and Polkovnichenko, 2005) can well be compatible with equilibrium. If an investor has superior information then underdiversification can even be necessary to outperform the market.

In our model alpha-opportunities can be explained as a feature of financial market equilibria. The larger the deviation of average expected returns from true returns the higher the alpha-opportunities. Moreover we can show that alpha-opportunities erode with the assets under management, which is a feature that has been observed for many active portfolio managers, as for example for hedge funds (cf. Getmansky, 2012; and Agarwal, Daniel, & Naik, 2009). In our model this important feature has a very simple explanation. The more wealth a strategy acquires the more it determines the market portfolio since the latter is the wealth weighted average of the individual portfolios. Thus the strategy will resemble the market portfolio, which, by definition, has an alpha of zero, because the alpha is proportional to the deviation of the individual expectations from the market expectations. Note that our model gives an equilibrium explanation of this feature that does not need to refer to any ad-hoc ideas of a production function for alpha opportunities (cf. Berk & Green, 2004). Moreover, in our model the hunt for alpha-opportunities is a zero-sum game. If some investor generates a positive alpha there must be some other investor earning a negative alpha. Hence the ease to generate alpha opportunities depends on the sophistication of the other investors in the market. This feature may explain why hedge funds could generate very high returns during the stock market bubble at the turn of the millennium in which many unsophisticated investors took active bets. After the bubble burst, many unsophisticated investors left the market due to frustration and hedge fund returns decreased.

Finally, in our model it turns out that a market in which some investors acquire information to be active while others get the average information for free from market prices cannot be a stable outcome. Moreover, all investors being passive may also not be an outcome that is stable with respect to information acquisition if the average expectation is far from the true returns. Only if all investors are endowed with correct expectations then nobody needs to acquire better information and the market prices do not reveal better information for free. This is the case in the standard CAPM with homogenous beliefs. This result relates to the well know result of Grossman and Stiglitz (1980) on the impossibility of informationally efficient markets. However, in their model, to get a different solution to the lucky case in which all investors are initially endowed with correct beliefs, Grossman and Stiglitz (1980) introduce noise traders that reduce the informational efficiency of prices so that traders who acquire information get rewarded for it.

Our results give a common framework for many phenomena that have been discussed in the literature. Besides being able to address alpha-opportunities in a simple equilibrium framework, we can explain underdiversification, the erosion of alpha-opportunities as assets under management increase, and the structure of performance fees for active management. Moreover, our simple model gives a foundation of more applied research on active management like the one of Grinold and Kahn (2000) and Black and

Litterman (cf. Litterman, 2003). Our model provides a common ground for these two approaches whose methodologies seem to be in contradiction. While Grinold and Kahn (2000) argue for active portfolio management based on the mean-variance framework of Markowitz (1952), Black and Litterman argue for active portfolio management based on the security market line. According to Grinold and Kahn (2000) investors need to form their return expectations independently from other investors and then solve the mean-variance optimization problem suggested by Markowitz (1952) while Black and Litterman (cf. Litterman, 2003) suggest to first recover the market expectations from the SML and then to take positions relative to this (over- and underweighting of assets relative to the market portfolio). Black and Litterman assume that the security market line is a "center of gravity" towards which the financial markets tend over time. Hence an active Black-Litterman investor goes short in those assets that have realized a positive alpha because he infers from this that in the next period the return will most likely be decreasing. Our model gives support to this view since taking account for the optimal information acquisition in the long run all alphas will erode. Our approach can also accommodate active portfolio management in the sense of Grinold and Kahn (2000). As we show below, optimal mean-variance portfolios must lie on a security line which is the security market line in which market expectations have been replaced by individual expectations. The security market line is then obtained by the aggregation of these individual security lines. An active mean-variance investor à la Grinold and Kahn "sees" alpha opportunities because he holds a belief of expected returns that deviates from the average belief of the investors expressed in the security market line.

Of course we do not claim that our simple model can explain all features of active management. In particular some features related to hedge funds, as for example higher order returns, lead out of the mean-variance framework. However, since a simple CAPM with heterogenous beliefs carries us quite far in the understanding of many important features of active management this framework can give a first intuition for what active management is about.

3. A CAPM with Heterogenous Beliefs

We consider a one period financial market model spanning from t = 0 to t = 1. There are K assets, $k = 0, 1, \ldots, K$, traded in t = 0 and having payoffs in t = 1. Asset $k = 0$ is riskless and its return is denoted by R_f. Assets $k = 1, \ldots, K$, are risky with return $R_k, k = 1, \ldots, K$. Without loss of generality we normalize the supply of all risky assets to 1. By $\mu_k = \mathbb{E}(R_k)$ we denote the expected return of asset $k, k = 1, \ldots, K$, and by $COV = (COV(R_k, R_l))_{k,l=1,\ldots,K}$ we denote the covariance matrix of asset returns.

There are I investors. Investor i has initial wealth $w^i > 0$ and mean-variance preferences over date 1 returns

$$U^i(\mu, \sigma) = \mu - \frac{\gamma^i}{2}\sigma^2,$$

where $\gamma^i > 0$ measures investor i's risk aversion and μ and σ are the expected return and variance, respectively, of investor i's portfolio. We assume that investors do not know the distribution of asset payoffs but rather hold individual beliefs over expected asset returns and the covariance matrix of asset returns.

Let μ_k^i denote i's belief about expected return of asset k and let COV^i denotes i's belief about the covariance matrix of returns. Using the budget constraint $\sum_{k=0}^{K} \lambda_k = 1$ we can express the portfolio optimization problem as a maximization problem in the allocation to risky assets[3] as follows. Given the portfolio of risky assets λ^i investor i invests $\lambda_0^i = 1 - \sum_{k=1}^{K} \lambda_k^i$ into the riskless asset.

Given her beliefs μ^i and COV^i investor i solves

[3] We denote by the vector $\lambda \in \mathbb{R}^K$ the column vector of allocations to the risky asset. When we refer to the allocation to the risk free asset we denote this by λ_0.

$$\max_{\lambda \in \mathbb{R}^K} \lambda^T \left(\mu^i - R_f e\right) - \frac{\gamma^i}{2} \lambda^T COV^i \lambda, \tag{1}$$

where $e \in \mathbb{R}^K$ is a column vector with all entries equal to 1. Thus the allocation of risky assets is chosen such that the mean, $\lambda^T\left(\mu^i - R_f e\right)$, minus the risk aversion, γ^i, times the variance, $\lambda^T COV^i \lambda$, of the portfolio is maximal. Since the expected return of the risky assets is higher than the risk free rate, the mean of the portfolio return increases in the allocation to risky assets. On the other hand this increases the portfolio variance.

The necessary and sufficient first order condition for the solution λ^i of (1) is

$$COV^i \lambda^i = \frac{\mu^i - R_f e}{\gamma^i}. \tag{2}$$

Thus the portfolio of risky assets is given by the inverse of the covariance matrix multiplied with the vector of excess returns.

Let q_k be the price of asset *k*. Since the supply of each asset is normalized to 1, in equilibrium $\frac{\sum_i w^i \lambda_k^i}{q_k}$ equals 1 which implies that q_k is equal to the market capitalization of asset *k*, i.e. in equilibrium we have

$$q = \sum_i w^i \lambda^i.$$

Hence, from the agent's optimal portfolio choice (2) we obtain that

$$q = \sum_i w^i (COV^i)^{-1} \frac{\mu^i - R_f e}{\gamma^i}.$$

From now on we assume that in equilibrium $\sum_{k=1}^K \lambda_k^i > 0$, so that $\lambda_0^i < 1$ for all *i*. This assumption means that investors should not hold a portfolio of risky assets with so many short positions, i.e. $\lambda_k^i < 0$, so that $\sum_{k=1}^K \lambda_k^i \leq 0$. Note that, however, investors can leverage as much as they like, i.e. they can choose $\lambda_0^i < 0$.

3.1. Security Market Line

In this section we derive the security market line for the CAPM with heterogenous expectations. In order to do so we need to specify how individual expectations are averaged to become the market expectation.

Let $w_f^i := (1 - \lambda_0^i) w^i$ be the financial wealth investor *i* invests into risky assets. By our assumption above $w_f^i > 0$ for all *i*. Let, accordingly,

$$r^i = \frac{w_f^i}{\sum_j w_f^j}$$

be the relative financial wealth invested by *i* and define

$$\rho := \left[\sum_i \frac{r^i}{\gamma^i (1 - \lambda_0^i)}\right]^{-1}$$

As will be shown next, ρ will be the decisive parameter in aggregating expectations. In our model it turns out that the appropriate aggregation rule is to define the average belief about the expected asset returns, $\bar{\mu}$, and the average belief about the covariance matrix of asset returns, $\overline{COV}$ as follows:

$$\bar{\mu} := \rho\overline{COV}\sum_i \frac{r^i}{\gamma^i(1-\lambda_0^i)}(COV^i)^{-1}\mu^i, \tag{3}$$

where

$$\overline{COV} := \frac{1}{\rho}\left[\sum_i \frac{r^i}{\gamma^i(1-\lambda_0^i)}(COV^i)^{-1}\mu^i\right]^{-1} \tag{4}$$

Thus the aggregated covariance has the structure of a harmonic mean of individual covariances and the aggregate expected returns are averages of individual returns combined with the covariances. Observe that under homogenous beliefs about the covariance matrix of asset returns, i.e. $COV^i = COV^*$ for all i, we obtain $\overline{COV} = COV^*$ and $\bar{\mu} = \sum_i a^i\mu^i$, where

$$a^i = \frac{w^i}{\gamma^i}\left(\sum_j \frac{w^j}{\gamma^j}\right)^{-1},$$

i.e. every individual's belief enters the average belief proportional to the individual's wealth divided by his risk aversion.

If all investors invest according to their risky portfolio $\bar{\lambda}^i := \frac{1}{1-\lambda_0^i}\lambda^i$, then in equilibrium the market portfolio is

$$\lambda^M = \sum_i r^i\bar{\lambda}^i = \frac{1}{\sum_k q_k}q.$$

Accordingly, let $\bar{\mu}_M = \sum_k \lambda_k^M \bar{\mu}_k$ be the average belief about the expected return $R_M = \sum_k \lambda_k^M R_k$ of the market portfolio. Then we can state the Security Market Line Theorem for average expectations. It shows that the average expected excess return is proportional to the average expected market return with a proportionality factor β, that is given by the average covariance of an assets return with the market return, normalized by the variance of the market return. Thus except for taking averages the security market line has the same structure as in the CAPM with homogenous expectations.

Proposition 3.1 (Security Market Line for Average Expectations)

In equilibrium the risk premium of any asset k is proportional to the risk premium of the market portfolio under average expectations, $\bar{\mu}$, as defined in (3). The factor of proportionality is given by the covariance of the return of asset k with the market portfolio divided by the variance of the market portfolio, where covariances and variances are determined with respect to $\overline{COV}$, as defined in (4):

$$\bar{\mu}_k - R_f = \underbrace{\frac{\overline{COV}(R_k,R_M)}{\bar{\sigma}^2(R_M)}}_{\beta_{M,K}}(\bar{\mu}^M - R_f), k = 1,\dots,K, \tag{5}$$

where $\overline{COV}(R_k,R_M) = \sum_{l=1}^K \lambda_l^M\,\overline{COV}_{k,l}$ *and* $\bar{\sigma}^2(R_M) = (\lambda^M)^T\overline{COV}\lambda^M$.

Proof: For all investors i we let $\bar{\lambda}^i$ denote *i*'s portfolio of risky assets, i.e.

$$\bar{\lambda}_k^i := \frac{\lambda_k^i}{1-\lambda_0^i} \text{ for all } k = 1,\dots,K.$$

We can then rewrite (2) to obtain

$$COV\bar{\lambda}^i = \frac{\mu^i - R_f e}{\gamma^i(1-\lambda_0^i)}. \tag{6}$$

From (6) it follows that

$$\lambda^M = \sum_i r^i \bar{\lambda}^i = \sum_i \frac{r^i}{\gamma^i\left(1-\lambda_0^i\right)}\left(COV^i\right)^{-1}\left(\bar{\mu} - R_f e\right)$$
$$= \frac{1}{\rho}\overline{COV}^{-1}\left(\bar{\mu} - R_f e\right)$$

Hence,

$$\bar{\mu} - R_f e = \rho\overline{COV}\lambda^M, \tag{7}$$

which implies that

$$\bar{\sigma}^2(R_M) = (\lambda^M)^T\overline{COV}\lambda^M = \frac{1}{\rho}\left(\bar{\mu}^M - R_f e\right). \tag{8}$$

Substituting (8) into (7) yields

$$\bar{\mu} - R_f e = \frac{\overline{COV}\lambda^M}{\bar{\sigma}_M^2}\left(\bar{\mu}^M - R_f e\right),$$

which proves the proposition. □

In the special case, where all investors have homogenous and correct beliefs about the covariance matrix of asset returns, i.e. $COV^i = COV$ for all *i*, the security market line for average expectations (5) reads

$$\bar{\mu}_k - R_f = \underbrace{\frac{COV(R_k, R_M)}{\sigma^2(R_M)}}_{\beta_{M,k}}\left(\bar{\mu}_M - R_f\right), k = 1, \dots, K, \tag{9}$$

A similar aggregation result as in Proposition 3.1 has been obtained by Chiarella *et al.* (2006). While we consider a *distribution economy*, where there is an exogenously given income distribution among investors as well as an exogenously given supply of assets, Chiarella *et al*. study an exchange economy, where investors are endowed with a portfolio of assets.[4] Moreover, Chiarella *et al*. assume that investors have a linear mean-variance utility function over final wealth, while we assume that they have a linear mean-variance utility function over returns. This difference is crucial as it has implications for the comparative statics of portfolios with respect to wealth: In the model of Chiarella *et al*. the portfolios of risky assets held by the investors are independent of wealth, i.e. if wealth increases, then all additional wealth is invested into the riskless asset, which appears to be in conflict with observed investment behaviour. In contrast, the mean-variance utility function we consider yields a fixed mix portfolio, i.e. the share of wealth invested into a risky asset is independent of wealth. Taking investors' beliefs as given, Chiarella *et al*. focus on an analysis of the impact of the diversity of heterogenous beliefs on equilibrium prices and trading volume. In this paper we will go a step further and study which beliefs will survive in the long run. Hence, the degree of heterogeneity will be endogenous in our model.

[4]For the difference between distribution and exchange economies cf. Malinvaud (1972).

Equation (5) is the security market line (SML) we obtain from aggregation of individual beliefs. This SML can be "seen" by an outside observer. An individual investor i, however, does not observe this SML. She sees an individual security market line defined with respect to her optimal portfolio of risky asset $\bar{\lambda}^i$ and her beliefs μ^i and COV^i. Let $R^{\bar{\lambda}^i} = \sum_l \bar{\lambda}_l^i R_l$ be the return of investor *i*'s portfolio of risky assets and let $\mu^i(R^{\bar{\lambda}^i}) = \sum_k \bar{\lambda}_k^i \mu_k^i$ be the expected return of her portfolio under her belief μ^i. Multiplying both sides of (6) with $\bar{\lambda}^i$ yields

$$\gamma^i(1-\lambda_0^i) = \frac{\mu^i\left(R^{\bar{\lambda}^i}\right) - R_f}{\sigma^2(R^{\bar{\lambda}^i})},$$

where $\bar{\sigma}^2\left(R^{\bar{\lambda}^i}\right) = (\bar{\lambda}^i)^T COV^i \bar{\lambda}^i$. Substituting this into (6) we obtain the individual SML of investor *i*:

Proposition 3.2 (Individual Security Market Line)

For any investor i the risk premium of any asset k is proportional to the risk premium of his portfolio, where the factor of proportionality is given by the covariance of the return of asset k with investor i's portfolio divided by the variance of i's portfolio and risk premia are determined according to μ^i*:*

$$\mu_k^i - R_f = \underbrace{\frac{COV^i\left(R_k, R^{\bar{\lambda}^i}\right)}{\sigma^{i^2}\left(R^{\bar{\lambda}^i}\right)}}_{\beta_k^i}\left(\mu^i\left(R^{\bar{\lambda}^i}\right) - R_f\right), k = 1, \dots, K, \tag{10}$$

where $COV^i\left(R_k, R^{\bar{\lambda}^i}\right) = \sum_l \lambda_l^i (COV^i)_{k,l}$ *and* $\sigma^{i^2}\left(R^{\bar{\lambda}^i}\right) = (\bar{\lambda}^i)^T COV^i \bar{\lambda}^i$.

The individual SML is thus a representation of the optimal portfolio choice. If the investor has chosen a mean-variance efficient portfolio all expected returns lie on his SML. In particular we see that investor i will hold the market portfolio if $\mu^i = \bar{\mu}$ and $COV^i = \overline{COV}$, i.e. if her beliefs coincide with the average beliefs in the market. To phrase this differently, if the market portfolio is mean-variance efficient, then all assets' expected returns are alingned on the SML.

As a final point of this section we show that in the CAPM with heterogenous expectations portfolios are typically underdiversified which is in line with considerable empirical evidence (cf. Odean, 1999; Goetzmann & Kumar, 2008; and Polkovnichenko, 2005). In the CAPM with heterogenous beliefs underdiversification is consistent with optimal investment. We illustrate this with the following simple example:

Example 3.1

Let there be two investors $i = 1, 2$, and two risky assets $k = 1, 2$. Let the covariance matrix of asset returns be given by

$$COV = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix},$$

where $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$. Moreover, assume that investor *i*'s belief about expected asset returns is given by

$$\mu^1 = \begin{pmatrix} d \\ R_f \end{pmatrix} \text{ and } \mu^2 = \begin{pmatrix} R_f \\ d \end{pmatrix},$$

where $d > R_f$. Then, it is straightforward to show that

$$\bar{\lambda}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \bar{\lambda}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, investor 1 invests only into asset 1 and investor 2 only invests into asset 2, while the market portfolio is given by

$$\lambda^M = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}.$$

Thus, in equilibrium each investor is underdiversified compared to the market portfolio.

4. The Alpha

The "alpha" is one of the most used terms in finance. It measures the deviation of mean asset returns from the security market line. Investment funds, in particular hedge funds, claim to generate a positive alpha in order to attract assets under management. Whether active funds like hedge funds have alpha is one of the most controversial debates in academia and also in practice. In an early summary of the debate Fama (1970) finds that on average active funds have no alpha. Many studies followed up on this seminal paper. The most recent one is from Barclays Capital (2016) which still comes to the same conclusion.

As we will show in the following, our model of a CAPM with heterogenous beliefs can explain the existence of a nonzero alpha for individual active investors. However, in the next section we will demonstrate that the alpha is not an appropriate performance measure when it comes to the choice between active and passive investment. Moreover, we will show that a weighted average of the individual alphas is zero, i.e. the hunting for alpha is a zero-sum game so that on average there is no alpha.

Under heterogenous beliefs there are several ways to define the alpha of a portfolio of risky assets. One important distinction of the possible alphas that can be defined is whether one takes an ex-ante or an ex-post point of view. We begin with the ex-ante point of view. As we showed above, every investor chooses his portfolio so that the expected returns lie on his individual security market line. With heterogenous beliefs the combination of points given by the individual expected returns and the market beta, $(\mu_k^i, \beta_{M,k}), k = 1, \ldots, K$, need however not lie on the SML. We call the distance $\mu_k^i - R_f - \beta_{M,k}(\mu_{M,k} - R_f)$ the ex-ante alpha that investor i expects to get relative to the SML and denote it by $\alpha_{M,k}^i$. The alpha of the portfolio of investor i is then given by $\alpha^i = \sum_{k=1}^K \lambda_{M,k}\, \alpha_{M,k}^i$. The following proposition shows the zero-sum property of these alphas.

Propositon 4.1. (Zero Sum Game of Ex-Ante Alphas)

Defining the Alpha as the excess return that agent i sees in asset k over and above the return seen by the market, the weighted average of the individual investors' alphas is zero, where the weights are given as in the security market line., i.e. $\sum_{i=1}^I a^i \alpha^i = 0$, *where, as before,* $a^i := \frac{w^i}{\gamma^i}\left(\sum_j \frac{w^j}{\gamma^j}\right)^{-1}$ *for all i.*

Proof: Recalling the definition of the alphas we get

$$\sum_{i=1}^I a^i \sum_{k=1}^K \lambda_k^M \alpha_{k,M}^i = \sum_{i=1}^I a^i \sum_{k=1}^K \lambda_k^M \left((\mu_k^i - R_f) - \beta_{k,M}(\bar{\mu}^M - R_f) \right)$$
$$= \sum_{i=1}^I a^i \sum_{k=1}^K \lambda_k^M (\mu_k^i - R_f) - \sum_{i=1}^I a^i \sum_{k=1}^K \lambda_k^M \beta_{k,M}(\bar{\mu}^M - R_f).$$

And hence, by the weighting factors and the market returns we get what we claimed:

$$\sum_{i=1}^{I} a^i \alpha^i = (\bar{\mu} \quad - R_f) - \beta_M(\bar{\mu}^M - R_f) = 0. \ \square$$

The interpretation of Proposition 4.1. is that investors do not agree on the expected returns but still they agree that ex-ante not all of them can be right.

The second alpha is the ex-post alpha given by the deviation from the security market line, which is defined with respect to the true expected returns and covariances of returns, taking the market portfolio as a benchmark. This is the alpha considered in the finance industry and we will use it to study the optimal choice between active and passive investment. We define the (ex post) alpha of asset *k* by

$$\hat{\alpha}_k := \hat{\mu}_k - R_f - \hat{\beta}_{M,k}(\hat{\mu}^M - R_f),$$

where $\hat{\mu}^M := \sum_k \lambda_k^M \hat{\mu}_k$ is the true expected return of the market portfolio and $\hat{\beta}_{M,k} = COV(R_k, R_M)/\sigma^2(R_M)$ is the true beta of asset *k* with respect to the market portfolio.[5] If all investors have homogenous and correct beliefs, i.e. $\mu^i = \hat{\mu}$ and $COV^i = COV$ for all all *i*, then all investors hold the market portfolio and $\hat{\alpha}_k = 0$ for all *k* by Propositions 3.1 and 3.2. Hence, in the standard CAPM with homogenous and correct beliefs there is no portfolio which generates a positive alpha. By contrast, under heterogenous beliefs, there typically exist portfolios generating a positive alpha. To see this recall that in equilibrium

$$\bar{\alpha}_k := \bar{\mu}_k - R_f - \bar{\beta}_{M,k}(\bar{\mu}^M - R_f) = 0$$

for all *k* by Proposition 3.1. We conclude that if average beliefs differ from the truth ($\bar{\mu} \neq \hat{\mu}$ and/or $\overline{COV} \neq COV$), then typically there exists *k* such that $\hat{\alpha}_k \neq \bar{\alpha}_k = 0$ and hence there exists a portfolio of risky assets $\bar{\lambda}$, which generates a positive alpha, i.e. $\sum_k \bar{\lambda}_k \hat{\alpha}_k > 0$.

Thus, our CAPM model with heterogenous expectations can explain the existence of a nonzero alpha in equilibrium. However, it turns out that the hunt for alpha opportunities is a zero sum game and that alpha opportunities erode whenever the investor accumulates too much wealth in the economy. Moreover, we will argue that a positive alpha is not necessarily a good criterion for active portfolio management. Hence, our model on the one hand provides a thorough foundation for the alpha and on the other hand casts serious doubt on its use in practical financial decisions.

In order to derive these results we define the ex post or true alpha of investor *i*'s portfolio as

$$\hat{\alpha}^i := \sum_{k=1}^{K} \bar{\lambda}_k^i \hat{\alpha}_k,$$

Now we are in a position to prove:

Proposition 4.2 (Zero Sum Game of Ex-Post Alphas)

Defining the alpha as the return that agent i gets in excess of the market, the wealth weighted average of the individual investors' alphas is zero, i.e. in equilibrium

$$\sum_i w_f^i \hat{\alpha}^i = 0.$$

Proof: We obtain[6]

[5] The notation of adding a hat on variables does not mean – as it would in empirical papers – that these variables are estimated.

[6] All sums over assets run from $k = 1$ to K.

$$\sum_i w_f^i \hat{\alpha}^i = \sum_i r^i \left(\sum_j w_f^j\right) \sum_k \bar{\lambda}_k^i \hat{\alpha}_k$$
$$= \left(\sum_j w_f^j\right) \sum_k \hat{\alpha}_k \lambda_k^M$$
$$= \left(\sum_j w_f^j\right)\left(\hat{\mu}^M - R_f \sum_k \lambda_k^M - (\hat{\mu}^M - R_f)\sum_k \hat{\beta}_{M,k}\lambda_k^M\right) = 0. \ \square$$

Hence, since $w_f^i > 0$ for all i, an investor *i* can generate a positive alpha if and only if there is another investor *j* who generates a negative alpha. □

Next, we will address the question, how the alpha of an investor behaves if she accumulates more and more wealth, so that, in the limit, she holds all wealth in the economy. In practice it has been observed that alpha-opportunities erode with the assets under management (Getmansky, 2012; and Agarwal *et al.*, 2009). Hence, a fund which becomes too big deprives itself of generating a positive alpha. As we will show, in our model an investor has a zero alpha in the limit, when she has accumulated all the wealth of the economy. The intuition is straightforward: In the limit, an investor who has accumulated all the wealth, must hold the market portfolio which has an alpha of zero. To make this intuition precise, we let $\left((w^{i,n})_i\right)_n$ be a sequence of wealth profiles. Then, by $\lambda^{M,n}$ we denote the market portfolio under the wealth profile $(w^{i,n})_i$, i.e.

$$\lambda^{M,n} = \sum_i \bar{\lambda}_k^i r^{i,n}, k = 1, \dots, K,$$

where

$$r^{i,n} = \frac{(1-\lambda_0^i)w^{i,n}}{\sum_j (1-\lambda_0^j) w^{j,n}} \text{ for all } i.$$

By $R_{M,n}$ we denote the equilibrium return of the market portfolio under the wealth profile $(w^{i,n})_i$, i.e.

$$R_{M,n} = \sum_k \lambda^{M,n} R_k.$$

Finally let, $\hat{\mu}^{M,n}$ denote the expectation of $R_{M,n}$ under the true beliefs. Then, for all *k*, we let $\hat{\alpha}_k^n$ denote the alpha of asset *k* at the wealth profile $(w^{i,n})_i$, i.e.

$$\hat{\alpha}_k^n = \hat{\mu}_k - R_f - \beta_{M,k,n}\left(\hat{\mu}^{M,n} - R_f\right),$$

where $\beta_{M,k,n} = COV(R_k, R_{M,n})/\sigma^2(R_{M,n})$.

Proposition 4.3 (Erosion of Alpha Opportunities)

Let $\left((w_0^{i,n})_i\right)_n$ *be a sequence of wealth profiles such that*

$$\lim_{n\to\infty} \frac{w^{i,n}}{\sum_j w^{j,n}} = 1$$

for some i. Then

$$\lim_{n\to\infty} \hat{\alpha}^{i,n} = 0,$$

where

$$\hat{\alpha}^{i,n} = \sum_k \bar{\lambda}_k^{i,n} \hat{\alpha}_k^n \text{ for all } n.$$

Proof: From $\lim_{n\to\infty} w^{i,n}/\left(\sum_j w^{j,n}\right) = 1$ it follows that

$$\lim_{n\to\infty} r^{i,n} = 1,$$

which implies that

$$\lim_{n\to\infty} \lambda^{M,n} = \lim_{n\to\infty} \sum_j r^{j,n} \bar{\lambda}^j = \bar{\lambda}^i.$$

Hence,

$$\lim_{n\to\infty} R_{M,n} = \lim_{n\to\infty} \sum_k \lambda_k^{M,n} R_k = R^{\bar{\lambda}^i},$$

$$and \lim_{n\to\infty} \beta_{M,k,n} = \frac{COV(R_k, R^{\bar{\lambda}^i})}{\sigma^2(R^{\bar{\lambda}^i})}.$$

This implies

$$\lim_{n\to\infty} \hat{\alpha}_k^n = \lim_{n\to\infty} \left[\hat{\mu}_k - R_f - \frac{COV\left(R_k, R_{M,n}\right)}{\sigma^2(R^{M,n})}\left(\hat{\mu}^{M,n} - R_f\right)\right]$$
$$= \hat{\mu}_k - R_f - \frac{COV\left(R_k, R^{\bar{\lambda}^i}\right)}{\sigma^2(R^{\bar{\lambda}^i})}\left(\hat{\mu}\left(R^{\bar{\lambda}^i}\right) - R_f\right),$$

where $\hat{\mu}\left(R^{\bar{\lambda}^i}\right) = \sum_k \bar{\lambda}_k^i \hat{\mu}_k$. Hence,

$$\lim_{n\to\infty} \hat{\alpha}^{i,n} = \lim_{n\to\infty} \sum_{k=1}^{K} \bar{\lambda}_k^i \hat{\alpha}_k^n = 0$$

as claimed. □

5. Active and Passive Investment

In the previous section we have shown that a CAPM with heterogenous beliefs can explain the existence of a nonzero alpha. We have also seen that the hunt for alpha opportunities is a zero sum game and that an investor, who accumulates too much wealth, deprives himself of generating a positive alpha. The question we are going to address now is much more basic: Is alpha an appropriate performance measure, i.e. should investors base their investment decision on the alpha generated by a fund? In order to answer this question rigorously we have to look at investors' preferences. So the question is, whether an investor's utility is increasing in the alpha of the portfolio she holds. The main result of this section will answer this question in the negative.

In order to simplify the analysis from now on we will assume that all investors have homogenous and correct beliefs about the covariance matrix of asset returns. In the literature[7] the assumption of homogenous covariance expectations is frequently used and can be justified as many practitioners do

[7]The famous model of Brock and Hommes (1998), for example, is based on mean-variance optimizing agents that have heterogenous beliefs on expected returns but agree on covariances. This model does, however, only have one risky asset.

portfolio allocations using historic covariances while adjusting historic means to get reasonable expected returns. Also, as Chopra and Ziemba (1993) have shown mistakes in means hurt the investor more than equally sized mistakes in covariances. That is to say that heterogeneity in expected returns has a higher impact than heterogeneity in expected covariance.

Assumption (HCOV): $COV^i = COV \ for \ all \ i = 1, \dots, I.$

This assumption is innocuous as it is sufficient to falsify a hypothesis in a simple model. In our case the hypothesis is that an investor's utility is increasing in the alpha of the portfolio she holds. If this hypothesis is not true in a simple model, where all investors have homogenous beliefs about the covariances of asset returns, then it will not be true in a more general model. Moreover, as we have argued in the introduction, many practitioners do portfolio allocations using historic covariances, so that there is only heterogeneity in beliefs about expected returns.

In order to analyse the relation between alpha and investors' preferences we consider a particular decision problem, namely the choice between active and passive investment. There is considerable evidence that the share of active investment has been decreasing over time. Cremers and Petajisto (2009) find that between 1983 and 2003 there was a significant decline in the proportion of mutual funds that have a high active share, i.e. even the actively managed funds become more and more passive. A possible reason for this is that actively managed funds typically do not outperform passive investment in a stock market index while at the same time active funds impose high fees. The following analysis will provide a theoretical explanation for the fact that active investment in general does not outperform passive investment. As a consequence, in a stationary economy there will only be passive investment in the long run.

We study the choice between active and passive investment by letting investors choose whether to invest according to an individual belief, which is costly to obtain, or whether to invest according to the average belief, which can be observed without incurring any costs. More precisely, suppose that each investor *i* can generate her own belief μ^i about the expected return of the assets. Generating an individual belief reduces investor *i*'s return by $C^i > 0$. This cost can be interpreted as a cost for information acquisition or as a management fee imposed by an actively managed fund. If the investor does not invest in her own belief she observes the market belief, $\bar{\mu}$, without incurring any costs.

Let $\tilde{\mu}^i \in \{\mu^i, \bar{\mu}\}$ be investor *i*'s belief. If $\tilde{\mu}^i = \mu^i$, we call *i* an *active investor* and if $\tilde{\mu}^i = \bar{\mu}$, then *i* is called a *passive investor*. Recall that under homogenous beliefs about the covariances of asset returns,

$$\bar{\mu} = \sum_i a^i \tilde{\mu}^i, \tag{14}$$

where $a^i = \frac{w^i}{\gamma^i}\left(\sum_j \frac{w^j}{\gamma^j}\right)^{-1}$. Given the belief $\tilde{\mu}^i$, investor *i* optimally chooses

$$\lambda^i(\tilde{\mu}^i) := COV^{-1}\frac{\tilde{\mu}^i - R_f e}{\gamma^i},$$

and invests $1 - \sum_{k=1}^{K} \lambda_k^i\left(\tilde{\mu}^i\right)$ into the riskless asset. Hence, she obtains the portfolio return

$$R(\tilde{\mu}^i) := R_f + \sum_{k=1}^{K} \lambda_k^i(\tilde{\mu}^i)(R_k - R_f).$$

Clearly, a passive investor will hold the market portfolio λ^M of risky assets.

We assume that investors ex post observe the true expected returns $\hat{\mu}$.[8] We denote by $U^i_{\hat{\mu}}(\tilde{\mu}^i)$ investor i's ex post (experienced) utility under the true expected returns if she has invested according to the belief $\tilde{\mu}^i$, i.e.

$$U^i_{\hat{\mu}}(\tilde{\mu}^i) = \mathbb{E}\left(R(\tilde{\mu}^i)\right) - \frac{\gamma^i}{2}\sigma^2\left(R(\tilde{\mu}^i)\right).$$

Hence,

$$\begin{aligned} U^i_{\hat{\mu}}(\tilde{\mu}^i) &= R_f + \frac{1}{\gamma^i}\left(\tilde{\mu}^i - R_f e\right)^T COV^{-1}\left(\hat{\mu} - R_f e\right) \\ &- \frac{\gamma^i}{2}\left(\frac{\tilde{\mu}^i - R_f e}{\gamma^i}\right)^T COV^{-1}\left(\frac{\tilde{\mu}^i - R_f e}{\gamma^i}\right) \\ &= R_f + \frac{1}{\gamma^i}\left(\tilde{\mu}^i - R_f e\right)^T COV^{-1}\left(\hat{\mu} - \frac{1}{2}\tilde{\mu}^i - \frac{1}{2}R_f e\right). \end{aligned}$$

Observe that $U^i_{\hat{\mu}}(\tilde{\mu}^i)$ is maximized for $\tilde{\mu}^i = \hat{\mu}$, i.e. for the case, where i has correct beliefs. Investor i chooses $\tilde{\mu}^i = \mu^i$ if

$$U^i_{\hat{\mu}}(\mu^i) - C^i \geq U^i_{\hat{\mu}}(\bar{\mu})$$

and $\tilde{\mu}^i = \bar{\mu}$ otherwise.

We define the following scalar product on $\mathbb{R}^K$:

$$< x, y >:= x^T COV^{-1} y, \quad x, y \in \mathbb{R}^K. \tag{15}$$

Observe that $<\cdot, \cdot>$ is indeed a scalar product. In particular, $<\cdot, \cdot>$ is positive definite since COV and hence COV^{-1} is positive definite. Using $<\cdot, \cdot>$ we define the following norm on $\mathbb{R}^K$:

$$\| x \| := \sqrt{< x, x >} = \sqrt{x^T COV^{-1} x}, \quad x \in \mathbb{R}^K. \tag{16}$$

With respect to this norm, $U^i_{\hat{\mu}}(\mu)$ is decreasing in the distance of μ to $\hat{\mu}$ (the true expectations) as is shown in the following lemma. In particular the closer the expected returns are to the truth the higher is the utility.

Lemma 5.1 *Let* $\mu, \mu' \in \mathbb{R}^K$. *Then*

$$U^i_{\hat{\mu}}(\mu) - U^i_{\hat{\mu}}(\mu') = \frac{1}{2\gamma^i}\left(\| \hat{\mu} - \mu' \|^2 - \| \hat{\mu} - \mu \|^2\right).$$

Hence,

$$U^i_{\hat{\mu}}(\mu) > U^i_{\hat{\mu}}(\mu') \Leftrightarrow \| \hat{\mu} - \mu \| < \| \hat{\mu} - \mu' \|.$$

Proof:

$$\begin{aligned} U^i_{\hat{\mu}}(\mu) - U^i_{\hat{\mu}}(\mu') &= \frac{1}{\gamma^i}\left[< \mu - R_f e, \hat{\mu} - \frac{1}{2}\mu - \frac{1}{2}R_f e > - < \mu' - R_f e, \hat{\mu} - \frac{1}{2}\mu' - \frac{1}{2}R_f e >\right] \\ &= \frac{1}{\gamma^i}\left[< \mu, \hat{\mu} - \frac{1}{2}\mu > - < \mu', \hat{\mu} - \frac{1}{2}\mu' >\right] = \frac{1}{2\gamma^i}\left(\| \hat{\mu} - \mu' \|^2 - \| \hat{\mu} - \mu \|^2\right). \end{aligned}$$

From Lemma 5.1 it follows that investor i chooses $\tilde{\mu}^i = \mu^i$ if and only if

[8] The underlying idea is that investors do not revise their investment strategy frequently so that they get enough observations of the asset returns in order to get a very precise estimate of the true expected returns.

$$\| \bar{\mu} - \hat{\mu} \|^2 - \| \mu^i - \hat{\mu} \|^2 \geq 2C^i\gamma^i.^9 \tag{17}$$

□

The decision to become active or remain passive thus depends on the accuracy of the average belief, $\| \bar{\mu} - \hat{\mu} \|$, as well as on the accuracy of the investor's belief, $\| \mu^i - \hat{\mu} \|$. We say that $\| \bar{\mu} - \hat{\mu} \|$ measures the "efficiency of the market", while $\| \mu^i - \hat{\mu} \|$ measures the individual "skill" of investor *i*. Observe that the more efficient the market is, the smaller the distance of the average belief to the truth. Similarly, the more skilled an investor is, the closer is her belief to the truth. Hence, from (17) it follows that, ceteris paribus, investor i is more inclined to be passive the more risk averse she is, the lower her skill, the higher her investment cost and the more efficient the market is.

Recall that we set out in this paper to answer the question whether alpha is an appropriate performance measure. So the question is, whether the following equivalence holds:

$$\| \bar{\mu} - \hat{\mu} \| \gtreqless \| \mu^i - \hat{\mu} \| \Leftrightarrow \hat{\alpha}^i = \sum_{k=1}^{K} \bar{\lambda}_k^i \hat{\alpha}_k \gtreqless 0 \tag{18}$$

The following example shows that (18) does not hold in general. More precisely, the example demonstrates that $\hat{\alpha}^i$ can be positive although $\| \bar{\mu} - \hat{\mu} \| < \| \mu^i - \hat{\mu} \|$ so that investor i prefers to be passive at the given belief profile. Conversely, it is possible that $\hat{\alpha}^i$ is negative and $\| \bar{\mu} - \hat{\mu} \| > \| \mu^i - \hat{\mu} \|$ so that investor i prefers to be active if his costs C^i are sufficiently low. Thus alpha is not an appropriate measure for being active but one should be active only if the market is inefficient and one's beliefs are better than the average beliefs.

Example 5.1 Let $R_f = 1$ and let there be two risky assets. There are four investors $i = 1, 2, 3, 4,$ with the following characteristics:

$$\mu^1 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \mu^2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mu^3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mu^4 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\gamma^1 = \gamma^2 = \gamma^3 = \gamma^4 = 2$$

$$w^1 = w^2 = w^3 = w^4 = 10$$

COV is given by

$$COV = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and the true beliefs are

$$\hat{\mu} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Suppose now that all investors are active. We have $a^1 = a^2 = a^3 = a^4 = 1/4$ and hence

$$\bar{\mu} = a^1\mu^1 + a^2\mu^2 + a^3\mu^3 + a^4\mu^4 = \begin{pmatrix} 3 \\ \frac{11}{4} \end{pmatrix}.$$

We obtain

$$\| \hat{\mu} - \mu^1 \|^2 = \frac{17}{2},$$

[9] It is without loss of generality to assume that the investor chooses active investment if she is indifferent.

$$\| \hat{\mu} - \mu^2 \|^2 = \frac{1}{2},$$
$$\| \hat{\mu} - \mu^3 \|^2 = \frac{1}{2},$$
$$\| \hat{\mu} - \mu^4 \|^2 = 5,$$
$$\| \hat{\mu} - \bar{\mu} \|^2 = \frac{25}{32}.$$

Hence, investors 2 and 3 prefer to be active for sufficiently small costs C^2, respectively, C^3, while investors 1 and 4 prefer to be passive for all costs C^1, respectively C^4. The optimal portfolios of the investors (everyone is active!) are

$$\lambda^1 = \begin{pmatrix} \frac{5}{4} \\ 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \lambda^3 = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence,

$$\bar{\lambda}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \bar{\lambda}^2 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \bar{\lambda}^3 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \bar{\lambda}^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And the market portfolio is

$$\lambda^M = \sum_i r^i \bar{\lambda}^1 = \begin{pmatrix} \frac{8}{15} \\ \frac{7}{15} \end{pmatrix},$$

and hence

$$\beta_M = \frac{COV\lambda^M}{(\lambda^M)^T COV\lambda^M} = \begin{pmatrix} \frac{110}{113} \\ \frac{105}{113} \end{pmatrix}.$$

This implies

$$\hat{\alpha} = \hat{\mu} - R_f e - \beta_M\left(\hat{\mu}^M - R_f\right) = \begin{pmatrix} -\frac{7}{113} \\ \frac{8}{113} \end{pmatrix},$$

from which we compute

$$\hat{\alpha}^1 = \hat{\alpha}^T \bar{\lambda}^1 = -\frac{7}{113},$$

$$\hat{\alpha}^2 = \hat{\alpha}^T \bar{\lambda}^2 = -\frac{2}{113},$$

$$\hat{\alpha}^3 = \hat{\alpha}^T \bar{\lambda}^3 = \frac{3}{113},$$

$$\hat{\alpha}^4 = \hat{\alpha}^T \bar{\lambda}^4 = \frac{8}{113}$$

Hence, investors 1 and 2 generate a negative alpha by being active, but nevertheless, as we have seen above, investor 2 prefers to be active if her costs C^2 are sufficiently small. Moreover, investors 3 and 4 generate a positive alpha by active investment, but investor 4 prefers to be passive for all costs C^4. Thus the alpha being positive is neither a necessary nor a sufficient condition for improving an asset allocation.

We are now in the position to define the stability of a CAPM equilibrium under information acquisition. We say that a profile with heterogenous beliefs is stable if no investor wants to deviate from her decision whether to be active or passive:

Definition 5.1 *The profile* $\tilde{\mu} = (\tilde{\mu}^1, \ldots, \tilde{\mu}^I)$ *is* ***stable****, if the following condition is satisfied: For all i,*

$$\| \bar{\mu} - \hat{\mu} \|^2 - \| \mu^i - \hat{\mu} \|^2 \geq 2C^i\gamma^i \iff \tilde{\mu}^i = \mu^i. \tag{19}$$

As our definition of stability makes clear, investors do not take into account that their decision whether to be active or passive may change the average belief $\bar{\mu}$ and the true expected returns $\hat{\mu}$ since it may change the equilibrium price. One objection against our notion of stability might be that we seem to assume that investors know the true expected returns. However, all we require is that investors know how their own skill compares to the efficiency of the market which is something they may have learned from the past.

We will now characterize stable profiles of beliefs. To this end let $\tilde{\mu} = (\tilde{\mu}^1, \ldots, \tilde{\mu}^I)$ be some profile of beliefs. Then, from (14) it follows that expectations need to be averaged only over active investors.

$$\bar{\mu} = \left(\sum_{i:\tilde{\mu}^i = \mu^i} a^i \right)^{-1} \sum_{i:\tilde{\mu}^i = \mu^i} a^i \mu^i,$$

whenever $\{i: \tilde{\mu}^i = \mu^i\} \neq \emptyset$, and $\bar{\mu}$ is undetermined, i.e. arbitrary, otherwise. With this definition we can derive some surprising result that resembles the famous Grossman-Stiglitz (1980) information paradox according to which there is no equilibrium with costly acquisition of information when prices reveal that information for free. As a result it is impossible that prices reflect all available information.

Proposition 5.1 *There exists no stable profile* $\tilde{\mu}$ *where some investor is active, i.e.* $\tilde{\mu}^i = \mu^i$ *for some i.*

Proof: Suppose by way of contradiction that $\tilde{\mu}$ is stable and that $\{i: \tilde{\mu}^i = \mu^i\} \neq \emptyset$. Without loss of generality let $\{i: \tilde{\mu}^i = \mu^i\} = \{1, \ldots, J\}$. Then

$$\bar{\mu} = \frac{1}{\bar{a}_J} \sum_{j=1}^{J} a^j \mu^j,$$

where $\bar{a}_J := \sum_{j=1}^{J} a^j$. Without loss of generality let $\| \mu^1 - \bar{\mu} \| \leq \| \mu^2 - \bar{\mu} \| \leq \| \ldots \leq \| \mu^J - \bar{\mu} \|$. Then

$$\| \bar{\mu} - \hat{\mu} \| = \frac{1}{\bar{a}_J} \left\| \sum_{j=1}^{J} a^j (\mu^j - \hat{\mu}) \right\|$$

$$\leq \frac{1}{\bar{a}_J} \sum_{j=1}^{J} a^j \| \mu^j - \hat{\mu} \|$$

$$\leq \| \mu^J - \hat{\mu} \|$$

Hence, (19) is violated for $i = J$ contradicting the fact that $\tilde{\mu}$ is stable. □

Hence, we obtain the paradoxical result that there cannot be active investment in a stable market. The intuition is that the beliefs of active investors determine the average belief so that low-skilled investors prefer to free ride on the better beliefs of high-skilled active investors by investing passively according to the average belief. Proposition 5.1 therefore provides a theoretical explanation for the empirical observation that the share of active investment has been declining constantly over the last twenty years (cf. Cremers & Petajisto, 2009). Clearly, in reality we will always observe active investment as the economy is not stationary. In the language of our model non-stationarity corresponds to a change in the true belief $\hat{\mu}$. If the economy has settled in a stable situation, where there is only passive investment, then a shock to $\hat{\mu}$ may render active investment by a high-skilled investor profitable. Hence, temporarily, we will observe active investment. If then there is no new shock to $\hat{\mu}$ for some period of time, the economy will again settle in a stable situation with passive investment only until the next shock occurs.

Whether or not passive investment indeed leads to a stable situation depends on how $\bar{\mu}$, which is an arbitrary convention if all investors are passive, relates to the true beliefs $\hat{\mu}$: If the market is very "efficient", i.e. $\| \bar{\mu} - \hat{\mu} \|$ is close to zero, then (19) is violated for all *i*, so that every investor being passive ($\tilde{\mu}^i = \mu^i$ for all *i*) is stable. If, on the contrary, $\| \bar{\mu} - \hat{\mu} \|$ is large, so that there exists an investor *i*, for whom active investment is profitable, i.e. (17) is satisfied, then passive investment is not stable. In other words, the standard CAPM with homogenous beliefs $\bar{\mu}$ that are close to the true beliefs $\hat{\mu}$ according to the efficiency measure $\| \bar{\mu} - \hat{\mu} \|$, is the only stable outcome of our model.

Proposition 5.2 *The profile* $\tilde{\mu} = (\tilde{\mu}^1, \dots, \tilde{\mu}^I)$ *is stable if and only if there exists* $\bar{\mu}$ *such that*

(i) $\tilde{\mu}^i = \bar{\mu}$, *and*
(ii) $\| \bar{\mu} - \hat{\mu} \|^2 < 2C^i\gamma^i + \| \mu^i - \hat{\mu} \|^2$,

for all i.

Now we are in a position to address the structure of performance fees that are in line with the information acquisition decision of the investors. We have seen that there cannot be active investment in the long run. In the short run, however, in particular if the true belief $\hat{\mu}$ changes, there is a potential for active investment if the market is inefficient, i.e. $\| \bar{\mu} - \hat{\mu} \|$ is large and the skill is high, i.e. $\| \bar{\mu} - \hat{\mu} \|$ is small. Suppose now that an investor cannot invest actively on his own but has to invest into a fund if he wants to be active. This fund sells a portfolio λ which, from the perspective of investor *i*, corresponds to the belief

$$\mu^i = R_f e + \gamma^i COV\lambda,$$

which follows from (2). The question then is, how the fee of the fund should look like in order to induce the investor to invest into the fund.

From our previous analysis we obtain two conditions:

(iii) In order to give the fund manager the right incentives, the performance fee should be increasing in the skill of the manager, i.e. decreasing in $\| \mu^i - \hat{\mu} \|$, since $U^i_{\hat{\mu}}(\mu^i)$ is decreasing in $\| \mu^i - \hat{\mu} \|$.
(iv) In order for the investor to become active, the fee must be bounded above by a function that is decreasing in the risk aversion of the investor and in the efficiency of the market.

We get the following result:

Corollary 5.1 *Any performance-fee* $C^i = C^i(\| \mu - \hat{\mu} \|, \| \bar{\mu} - \hat{\mu} \|)$, *that is decreasing in* $\| \mu - \hat{\mu} \|$ *and that satisfies*

$$C^i \leq \frac{1}{2\gamma^i}(\| \bar{\mu} - \hat{\mu} \|^2 - \| \mu - \hat{\mu} \|^2),$$

fulfills the conditions (iii) and (iv).

Hence, the performance fee should reward the skill of the manager but should also discourage the manager to hunt for investment opportunities in efficient markets. Moreover, comparing agents with different degrees of risk aversion, we find that the more risk averse agents have a lower willingness to pay for active portfolio management and therefore are more inclined to be passive.

We have seen that only passive investment is stable. Nevertheless, in the short run, for example, due to changes in the exogenous uncertainty, some investors may find it profitable to become active. We will now show that active investment is profitable only if the investor's wealth is small relative to the aggregate wealth in the economy. In other words, profitable investment opportunities resulting from inefficient markets (i.e. $\| \bar{\mu} - \hat{\mu} \|$ large) erode if the investor accumulates too much wealth.

Proposition 5.3 *Let* $\left((w^{i,n})_i\right)_n$ *be a sequence of wealth profiles such that*

$$\lim_{n\to\infty} \frac{w^{i,n}}{\sum_j w^{j,n}} = 1$$

for some i. Then

$$\lim_{n\to\infty} \| \bar{\mu}^n - \hat{\mu} \| = \| \mu^i - \hat{\mu} \|,$$

where $\bar{\mu}^n = \sum_j a^{j,n} \tilde{\mu}^{j,n}$ *with* $\tilde{\mu}^{j,n} = \mu^i, \tilde{\mu}^{j,n} \in \{\mu^j, \bar{\mu}^n\}$ *for all* $j \neq i$, *and* $a^{j,n} = \frac{w^{j,n}}{\gamma^j}\left(\sum_h \frac{w^{h,n}}{\gamma^h}\right)^{-1}$ *for all j and all n.*

Proof: From $\lim_{n\to\infty} w^{i,n}/\left(\sum_j w^{j,n}\right) = 1$ it follows that $\lim_{n\to\infty} w^{j,n}/w^{i,n} = 0$ for all $j \neq i$. This implies

$$\lim_{n\to\infty} a^{i,n} = \lim_{n\to\infty} \frac{1}{\gamma^i}\left(\sum_j \frac{w^{j,n}}{w^{i,n}\gamma^j}\right)^{-1} = 1.$$

Hence,

$$\lim_{n\to\infty} \bar{\mu}^n = \lim_{n\to\infty}\left(\sum_{j:\tilde{\mu}^{j,n}=\mu^j} a^{j,n}\right)^{-1} \sum_{j:\tilde{\mu}^{j,n}=\mu^j} a^{j,n}\mu^j = \mu^i,$$

which implies that $\| \bar{\mu}^n - \hat{\mu} \| \to \| \mu^i - \hat{\mu} \|$. □

6. Conclusion

The main contribution of this paper is twofold. Firstly, our model of a CAPM with heterogenous beliefs provides a general equilibrium foundation for the alpha which is heavily used by the finance industry as an indicator for profitable investment opportunities. It turns out that alpha-opportunities erode with the assets under management and that the hunt for alpha-opportunities is a zero-sum game. Secondly, we have demonstrated that in our model the sign or size of alpha does not deliver an appropriate criterion for investment decisions. Instead, we have shown that the choice between active and passive investment should be based on a measure of the distance between the individual, respectively average belief and the true expected returns of the assets.

In addition, our paper contributes to the ongoing discussion about the underperformance of active investment by showing that as long as there are active investors in the market, at least one active investor will prefer to become passive. Hence, our model predicts the market share of passive investment to grow over time. This is consistent with the empirical observation that even actively managed funds have become more and more passive over time.[10]

Our model is purely static. In particular, we have assumed that investors have correct expectations about the quality of their beliefs in terms of the distance to the true beliefs. An interesting topic for future research would be to study a dynamic version of our model, where in each period investors can choose between active and passive investment and where they learn about the quality of their beliefs or may even adjust their beliefs over time.

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